Julia sets in small copies of the Mandelbrot set

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Abstract

We will be interested in these few pages to the understanding of the different shapes Julia sets can have inside small copies of the Mandelbrot set. First, I will need to introduce tools necessary to do what is called quasiconformal surgery in low dimensional holomorphic dynamics. These will be necessary to define later notions and to do some prooves. After that, I will introduce basic notions of complex dynamics and some properties of the Julia sets and their connectedness loci, especially for quadratic polynomials. In the next section, I present the main interest of this report based on an article of Haïssinsky [Haï00] where he constructs straightening maps from small copies of the Mandelbrot sets into the whole set, and gives the exact shape of the Julia sets of such families of polynomials. And I end by providing an incomplete construction of the inverse of straightening maps based on the result of the previous section using two different methods.

Contents

Notations

- $\mathbb{C}$: The complex plane.
- $\widehat{\mathbb{C}}$: The Riemann sphere.
- $\mathbb{A}_{r,R}$: The standard annulus bounded by the circles of rays r and R.
- $\mathbb{A}_r$: The standard annulus bounded by the circles of rays r and 1.
- μ: Beltrami coefficient.
- σ: Almost complex structure.
- σ_0: The standard almost complex structure of $\mathbb{C}$.
- F_f (resp. F_c).: The Fatou set of f (resp. Q_c).
- J_f (resp. J_c): The Julia set of f (resp. Q_c).
- K_f (resp. K_c): The filled Julia set of the polynomial f (resp. Q_c).
- R_θ (resp. $R_c(\theta)$): The external ray of angle θ (resp. for Q_c).
- $\mathscr{M}$: The Mandelbrot set.
- $\mathscr{M}_{c_0}$: The copy of the Mandelbrot set around the hyperbolic component of center c_0.

1 Quasiconformal tools

Holomorphic maps are very rigid and do not paste into holomorphic mappings due to the property of analytic continuation. And pasting can be a powerful tool to create new holomorphic dynamics, this section will provide the weaker notion of quasiconformal maps which are much more flexible and provide, thanks to the Integrability theorem, holomorphic maps which are topologically conjugated to these maps and thus have the same dynamical properties.

Almost all proofs of the theorem and propositions stated in this section and especially those of the different equivalent definition of quasiconformal mappings and that of the integrability theorem are present in Ahlfors lectures [Ahl06]. The other proofs are either in [BF14] or in [Hub06] which introduce more useful properties of the quasiconformal theory when applied to holomorphic dynamics.

1.1 Quasiconformal Geometry

Before defining Quasiconformal mappings, we need to introduce some basic geometric notions in the complex plane first.

Definition 1.1.1 (Beltrami coefficient and Dilatation)**.** Let $L : \mathbb{C} \to \mathbb{C}$ be an invertible and orientation preserving $\mathbb{R}$-linear map of the complex plane. There exists $a, b \in \mathbb{C}$ such that

$$\forall z \in \mathbb{C}, \quad L(z) = az + b\bar{z}, \quad \textit{and} \quad |a| > |b|.$$

We define *the Beltrami coefficient* of L as $\mu(L) = \frac{a}{b}$.
We define *the dilatation* of L as $K(L) = \frac{1+|\mu|}{1-|\mu|} = \frac{|a|+|b|}{|a|-|b|}$.

Interpretation. Using the same notations as in the definition, let $E(L)$ be the inverse image of the unit circle by L. If we denote by $\theta \in \mathbb{R}/\mathbb{Z}$ such that $\mu(L) = |\frac{a}{b}|e^{i2\theta}$, we get $E(L)$ is the ellipse with half major axis $\frac{1}{|a|(1-|\mu|)}$ along the direction $e^{i(\theta+\frac{\pi}{2})}$, and half minor axis $\frac{1}{|a|(1+|\mu|)}$ along the direction $e^{i\theta}$ as we see in figure **??**.
Conversely, if we have an ellipse E of half major and minor axis M and m respectively, and θ the argument of the minor axis, then the Beltrami coefficient of the associated linear map L is given by $\mu(L) = \frac{M-m}{M+m}e^{i2\theta}$.
Therefore, the Beltrami coefficient characterises the ellipse $E(L)$.

The dilatation $K(L)$ is seen as the ratio of the major axis to the minor axis. It determines the shape of the ellipse up to scaling but not the position of its axes.

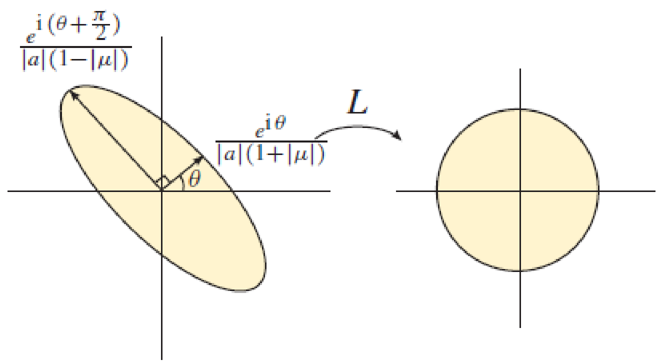


Figure 1: Ellipse determined by L, from [BF14]

Remark.

- One can notice that L is holomorphic iff its Beltrami coefficient is 0.
- (Fundamental) Giving a complex structure σ to the plane $\mathbb{R}^2$ is giving an element I and a multiplication such that $I^2 = -1$. And every linear map L gives a new complex structure to the plane by conjugation. This structure depends only on the ellipse $E(L)$. Hence every ellipse defines a complex structure on the plane.

The remark above leads to the following notion.

Definition 1.1.2 (Almost complex structure)**.** Let $U \subset \mathbb{C}$ and open domain. An *Almost complex structure* on U is a measurable field of infinitesimal ellipses $\mathscr{E} \in TU$ where TU is the tangent bundle of U.

This is equivalent to saying that for each $u \in U$, one defines an ellipse $E_u \subset T_uU$ such that the function $u \to \mu(u)$ from U to $\mathbb{D}$ giving the Beltrami coefficient is measurable with respect to the Lebesgue measure.

Every ellipse E_u gives a complex structure $\sigma(u)$ to T_uU. And the almost complex structure is denoted by σ.

We define the dilatation of σ by $K(\sigma) = \underset{u\in U}{esssup} K(u) \in [1, +\infty]$ where $K(u)$ is the dilatation of E_u.

We may construct new complex structures using mappings f satisfying some properties.

Definition 1.1.3. Let $U, V \subset \mathbb{C}$. We denote by $D^+(U, V)$ the class of continuous orientation preserving mappings $f : U \to V$ which are almost everywhere $\mathbb{R}$-differentiable and with a non-singular differential $D_uf : T_uU \to T_uV$ almost everywhere, depending measurably on u.

The differential of f at $u \in U$ given by $D_u f = \partial_z f(u)dz + \partial_{\bar{z}} f(u)d\bar{z}$ defines an infinitesimal ellipse in $T_u U$ of Beltrami coefficient $\mu_f(u) = \frac{\partial_{\bar{z}} f}{\partial_z f}$. This defines an almost complex structure σ_f on U with Beltrami coefficient μ_f, called *the pullback* of σ_0 by f. We denote $\sigma_f = f^*\sigma_0$ and $\mu_f = f^*\mu_0$.

Remark. One can generalize the notion of pullback to any initial almost complex structure σ instead of σ_0. But this requires the mapping to have an additional property so that the final structure is defined on a full measure set:
Let $D_0^+(U,V)$ denote the subclass of $D^+(U,V)$ consisting of absolutely continuous mappings with respect to the Lebesgue measure, that is the preimage of a zero measure set is another zero measure set.
Let $f \in D_0^+(U,V)$ and σ an almost complex structure on V with Beltrami coefficient μ. For each $v \in V$, the preimage E_u of E_v defines a complex structure $\sigma_f(u)$ in $T_u U$ for $u = f^{-1}(v)$. This defines an almost complex structure $\sigma_f = f^*\sigma$.
If the inverse map f^{-1} is absolutely continuous, we can define the pushforward $f_* = (f^{-1})^*$. In this case, $K_f = K_{f^{-1}}$.

Proposition 1.1.1. *Let $U, V, W \subset \mathbb{C}$, $f \in D_0^+(U,V)$, $g \in D_0^+(V,W)$ and μ a Beltrami coefficient on V. Then, we have:*

(i)

$$\forall u \in U, \quad f^*\mu(u) = \frac{\partial_{\bar{z}} f + \mu(f(u))\overline{\partial_z f(u)}}{\partial_z f(u) + \mu(f(u))\overline{\partial_{\bar{z}} f(u)}} \tag{1}$$

(ii) If f is holomorphic,

$$\forall u \in U, \quad f^*\mu(u) = \mu(f(u))\frac{\overline{\partial_z f(u)}}{\partial_z f(u)} \tag{2}$$

(iii)

$$\forall u \in U, \quad \mu_{g\circ f}(u) = \frac{\mu_f(u) + \mu_g(f(u))e^{-i2arg(\partial_z f(u))}}{1 + \mu_f(u)\mu_g(f(u))e^{-i2arg(\partial_{\bar{z}} f(u))}} \tag{3}$$

(iv)

$$K_{g\circ f} \leq K_f K_g$$

Definition 1.1.4 (f-invariant almost complex structure)**.** Let $U \subset \mathbb{C}$ be an open subset, and let $f \in D_0^+(U,U)$. Let σ be an almost complex structure on U. σ is *an f-invariant almost complex structure* if $f^*\sigma = \sigma$. f is said to be holomorphic or equivalently conformal with respect to σ.

Remark. Note that for any holomorphic map $f: U \to U$, $f^*\sigma_0 = \sigma_0$.

1.2 Quasiconformal mappings

There are several non trivially equivalent definitions of quasiconformal mappings which provides many ways of interpretation for such a notion.

We start by presenting the analytic and less intuitive ones.

Definition 1.2.1 (Analytic definition of quasiconformal mappings)**.** Let $U, V \subset \mathbb{C}$ be open domains and let $K \geq 1$. Set $k = \frac{K-1}{K+1}$. Then $\phi : U \to V$ is K-quasiconformal iff

(i) ϕ is a homeomorphism;

(ii) ϕ belongs to the Sobolev space $W^{1,2}_{loc} = H^1_{loc}$;

(iii) $|\overline{\partial}\phi| \leq k|\partial\phi|$ in L^2_{loc}.

An equivalent definition is to replace the condition (ii) by ϕ being absolutely continuous on lines (ACL).

Definition 1.2.2 (ACL)**.** Let $f : I \subset \mathbb{R} \to \mathbb{C}$ be a continuous map. f is *absolutely continuous on* I if

$$\forall \epsilon > 0, \quad \exists \delta > 0, \quad such \quad that \quad \sum_j |f(b_j) - f(a_j)| < \epsilon$$

for all non-intersecting intervals (a_j, b_j) of I of total length $\sum_j |b_j - a_j| < \delta$.

Let $f : U \subset \mathbb{C} \to \mathbb{C}$ be a continuous map. f is *ACL* if for any family of parallel lines of a disk in U, f is absolutely continuous on almost every line of it.

In order to define a Beltrami form of a quasiconformal map, we need the following theorem.

Theorem 1.2.1. *Let $f : U \to V$ be a continuous open map. If f has partial derivatives almost everywhere, then it is $\mathbb{R}$-differentiable almorst everywhere.*

Remark. Note that this definition imposes $K_\phi < +\infty$.

For the geometric definitions of quasiconformal mappings, we need a preliminary definition for quadrilaterals and cylinders.

Definition 1.2.3 (Conformal modulus)**.** A *quadrilateral* $Q = Q(z_1, z_2, z_3, z_4)$ is a Jordan domain in $\mathbb{C}$ with an ordered sequence (z_1, z_2, z_3, z_4) in its boundary called *the vertices* of Q. There exists a Riemann mapping φ for Q to a rectangle sending vertices to vertices. The *conformal modulus* of Q is defined as the well-defined real

$$mod\, Q(z_1, z_2, z_3, z_4) = \frac{|\varphi(z_1) - \varphi(z_2)|}{|\varphi(z_2) - \varphi(z_3)|} \tag{4}$$

An *annulus* A is a doubly connected domain of $\widehat{\mathbb{C}}$. There exists $0 \leq r < R \leq +\infty$ unique up to multiplication such that A is conformal to the standard annulus $A_{r,R} = \{z \in \mathbb{C} \,|\, r \leq |z| \leq R\}$. We define the *conformal mofulus* of A as

$$mod\, A = mod\, A_{r,R} = \begin{cases} \frac{1}{2\pi} log \frac{R}{r} & if \quad r > 0 \quad and \quad R < +\infty \\ \infty & if \quad r = 0 \quad or \quad R = +\infty \end{cases} \tag{5}$$

for an orientation preserving homeomorphism $\phi : U \to V$, the *maximal dilatation* of ϕ is defined as

$$K_\phi = \sup_{Q\, quadri \subset U} \frac{mod\, \phi(Q)}{mod\, Q} \tag{6}$$

Definition 1.2.4 (Geometric definition of quasiconformal mappings)**.** Let $U, V \subset \mathbb{C}$ and $K \geq 1$. Then $\phi : U \to V$ is a K-quasiconformal mapping iff ϕ is an orientation preserving homeomorphism satisfying

$$\frac{1}{K} mod\, Q \leq mod\, \phi(Q) \leq K mod\, Q$$

for every quadrilateral Q compactly contained in U, that is

$$K_\phi \leq K$$

Equivalently, $\phi : U \to V$ is a K-quasiconformal mapping iff ϕ is an orientation preserving homeomorphism satisfying

$$\frac{1}{K} mod\, A \leq mod\, \phi(A) \leq K mod\, A$$

for every annulus A compactly contained in U.

We present some properties of quesiconformal mappings.

Proposition 1.2.1. *Let $\phi : U \to V$ be an orientation preserving homeomorphism and $K \geq 1$. Then, we have*

1. *If ϕ is K-quasiconformal, ϕ^{-1} is K-quasiconformal.*
2. *If ϕ is K-quasiconformal, its composition on the left or right by a conformal map is also K-quasiconformal.*
3. *for $K-1, K_2 \geq 1$, the composition of a K_1-quasiconformal map with a K_2-quasiconformal map is K_1K_2-quasiconformal.*
4. *ϕ is K-quasiconformal iff it is locally K-quasiconformal.*

Theorem 1.2.2. *If ϕ is a quasiconformal map, it maps zero measure sets to zero measure sets. Moreover, for every measurable set E, $\lambda(\phi(E)) = \int_E Jac(\phi) d\lambda$.*

Remark. The theorem above implies that it is possible to pullback and pushforward complex structures by quasiconformal maps.
It also implies that $\partial_z \phi \neq 0$ and $Jac(\phi) > 0$ *a.e.*

The following theorem is central in the application of quasiconformal surgery to Complex dynamics.

Theorem 1.2.3 (Weyl's lemma)**.** *If ϕ is 1-quasiconformal, then it is conformal. Equivalently, if ϕ is quasiconformal and $\partial_{\bar{z}} \phi = 0$ a.e, then ϕ is conformal.*
The lemma will be mostly used under the following form: if $\phi^* \sigma_0 = \sigma_0$, then ϕ is holomorphic.

We also have a compactness property

Theorem 1.2.4 (Compactness)**.** *The set of K-quasiconformal mappings on $\mathbb{D}$ fixing 0 is compact for the topology of uniform convergence on compact subsets.*

Another notion that generalizes that of quasiconformal mappings is the following.

Definition 1.2.5 (Quasiregular mappings)**.** Let $U \subset \mathbb{C}$ be an open domain and $K < \infty$.A mapping g is *K-quasiregular* if it is of the form $g = f \circ \phi$ where $\phi : U \to \phi(U)$ is K-quasiconformal and $f : \phi(U) \to g(U)$ is holomorphic.
Equivalently, g is K-quasiregular if it is locally K-quasiconformal except at a discrete set of points of U.

Remark.

1. All the transitivity properties remain true for quasiregular mappings.
2. Weyl's lemma remain true for quasiregular mappings.
3. The quasiconformal conjugate of a holomorphic map is quasiregular.

The interest of such a notion is the fact that it can also be used to perform quasiconformal surgery thanks to the following.

Proposition 1.2.2. *Quasiregular maps and their inverse branches send sets of zero measure to sets of zero measure. Consequently, the pullback of a Beltrami form defined a.e., by a quasiregular map, is well defined a.e.*

Remark. All these notions can be extended to Riemann surfaces in the natural way using charts. I will not precise further details in this report.

1.3 Integrability theorem

This subsection deals with the main theorem that makes quasiconformal mappings useful to find topologically conjugate holomorphic mappings.

Theorem 1.3.1 (Integrability theorem)**.** *Let S be a simply connected Riemann surface and σ be an almost complex structure on S with Beltrami measure μ. Suppose the dilatation of σ uniformly bounded i.e $K(\sigma) < +\infty$, or equivalently $k = ||\mu||_\infty < 1$. Then μ is integrable i.e there exists a quasiconformal mapping $\phi : S \to \widehat{S}$ where $\widehat{S}$ is equal to $\mathbb{D}, \mathbb{C}$ or $\widehat{\mathbb{C}}$ such that $\phi^*\mu_0 = \mu$.*
Moreover, ϕ is unique up to post-composition by an automorphism of $\widehat{S}$.

This theorem is also known as the *measurable Riemann mapping theorem.*

Theorem 1.3.2 (Dependence on parameters)**.** *Let Λ be an open subset of $\mathbb{C}^N$ for some $N > 1$. Let S be a simply connected Riemann surface and let $(\mu_\lambda)_{\lambda\in\Lambda}$ be a family of measurable Beltrami coefficients on S. Suppose $\lambda \to \mu_\lambda(s)$ is continuous (respectively differentiable, real analytic) in λ for each $s \in S$, and assume that there exists $k < 1$ such that for all $\lambda \in \Lambda$, $||\mu_\lambda||_\infty \leq k$. For an appropriate normalization, let $\phi_\lambda : S \to \widehat{S}$ be the unique quasiconformal map integrating μ_λ. Then, for every $s \in S$, the map $\lambda \to \phi_\lambda(s)$ is continuous (respectively differentiable, real analytic) in λ.*

Remark. If $\widehat{S} = \mathbb{C}$ or $\widehat{\mathbb{C}}$, the theorem works if we replace continuity by holomorphy in λ, but it is not the case for $\mathbb{D}$.
The dependence of the inverse maps ϕ^{-1} may not be continuous. But using the implicit functions theorem, we can proof that it is if all the maps are differentiable with respect to $s \in S$.

1.4 Boundary behaviour of quasiconformal maps

Boundary behaviour of quasiconformal and maps is deeply studied in [Pom92], but we will only need some basic theorems.

We start by introducing the notion that will characterize the boundary curves and their parametrization.

Definition 1.4.1 (Quasisymmetry)**.** Let $h : \mathbb{S}^1 \to \mathbb{C}$ be a map. h is said to be *M-quesisymmetric* for some $M \geq 1$ if it is injective and

$$\forall z_1, z_2, z_3 \in \mathbb{S}^1, \quad 0 \neq |z_1 - z_2| = |z_2 - z_3| \Rightarrow \frac{1}{M} \leq \frac{|h(z_1) - h(z_2)|}{|h(z_2) - h(z_3)|} \leq M \tag{7}$$

which can be written for $H(t) = h(ei2\pi t)$,

$$\forall x \in \mathbb{T}, \forall t > 0, \quad \frac{1}{M} \leq \frac{|H(x+t) - H(x)|}{|H(x) - H(x-t)|} \leq M \tag{8}$$

Equivalently, h is quasisymmetric if there exists a strictly increasing continuous function $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\forall z_1, z_2, z_3 \in \mathbb{S}^1, \quad \frac{1}{\lambda(\frac{|z_2 - z_3|}{|z_1 - z_2|})} \leq \frac{|h(z_1) - h(z_2)|}{|h(z_2) - h(z_3)|} \leq \lambda(\frac{|z_2 - z_3|}{|z_1 - z_2|}) \tag{9}$$

Proposition 1.4.1.

- *The inverse of a quasisymmetric map is also quasisymmetric.*
- *A well defined composition of quasisymmetric maps is also quasisymmetric.*
- *A $\mathscr{C}^1$-diffeomorphism is quasisymmetric.*

Definition 1.4.2 (Quasicurves)**.**

- A Jordan arc (resp. curve) γ in $\mathbb{C}$ is a *quasiarc* (resp. *quasicircle*) if there exists $C > 0$ such that for all $z_1, z_2 \in \gamma$, *diam* $\gamma(z_1, z_2) \leq C|z_1 - z_2|$ where $\gamma(z_1, z_2)$ is tha arc of smallest diameter of γ joining z_1 to z_2.

- A Jordan domain bounded by a quasicircle is a *quasidisc.*
- A connected domain bounded by two quasicircles is a *quasiannulus.*

Proposition 1.4.2. *If $h : [a, b] \to \mathbb{C}$ (resp. $h : \mathbb{S}^1 \to \mathbb{C}$) is quasisymmetric, then its image γ is a quasiarc (resp. quasicircle).*

Conversely, a quasiarc or a quasicircle is the image of an interval or the unit circle under some quasisymmetric map.

Remark. Quasiarcs do not have cusps, but they may be nowhere differentiable like the Koch snowflake which is a quasidisc unlike the cardioïd.

Theorem 1.4.1 (Extension of quasiconformal mappings)**.** *We have the following two extension properties:*

(i) Let $G \subset \mathbb{C}$ be a quasidisc and $f : \mathbb{D} \to G$ a quasiconformal map.Then f extends continuously to a quasisymmetric map $f : \mathbb{S}^1 \to \partial G$.

(ii) Let A be a quasiannulus with boundaries γ_0 and γ_1 and let $f : \mathbb{A}_r \to A$ be a quasiconformal homeomorphism. Then f extends continuously to the boundaries as quasisymmetric maps.

The result is sharper for conformal maps as stated in the following.

Theorem 1.4.2 (Extension of conformal mappings)**.** *Let $G \subset \mathbb{C}$ be a Jordan domain with boundary $\gamma = \partial G$ and $f : \mathbb{D} \to G$ a conformal isomorphism. Then:*

1. If γ is $\mathscr{C}^{n+1}$ for some $n \geq 0$ then the n-th first derivatives of f extend continuously to $\overline{\mathbb{D}}$.

2. γ is analytic iff f extends conformally to a neighbourhood of $\overline{\mathbb{D}}$.

Definition 1.4.3 (More general definition of quasisymmetry)**.** Let G_1 and $G - 2$ two quasidiscs with boundaries γ_1 and γ_2 respectively and let $\mathscr{R}_i : \mathbb{D} \to G_i$ the corresponding Riemann mappings extended to the boundaries by the parametrizations $\mathscr{R}_i : \mathbb{S}^1 \to \gamma_i$ given by the extension theorem 1.4.1. An orientation preserving homeomorphism $f : \gamma_1 \to \gamma_2$ is *quasisymmetric* if $f \circ \mathscr{R}_1 : \mathbb{S}^1 \to \gamma_2$ is quasisymmetric, and equivalently if $\mathscr{R}_2^{-1} \circ f \circ \mathscr{R}_1 : \mathbb{S}^1 \to \mathbb{S}^1$ is quasisymmetric.

Now we state the interpolation theorems which are proved in the chapter 2 of [BF14].

Theorem 1.4.3 (Interpolation for quasidiscs and quasiannuli)**.** *We have the two following properties:*

(i) Let G_1 and G_2 two quasidiscs with boundaries γ_1 and γ_2 respectivelyn and let $f : \gamma_1 \to \gamma_2$ be a quasisymmetric map. Then f extends to a quasiconformal map $f : \overline{G_1} \to \overline{G_2}$.

(ii) For $j = 1, 2$, let A_j be a quasiannuli bounded by quasidiscs γ_j^i and γ_j^o. Let $f^i : \gamma_1^i \to \gamma_2^i$ and $f^o : \gamma_1^o \to \gamma_2^o$ be two quasisymmetric maps. Then there exists a quasiconformal map $f : \overline{A_1} \to \overline{A_2}$ extending the boundary maps f^i and f^o.

Remark. If the quasisymmetric maps are replaced by $\mathscr{C}^n$ maps, then the resulting interpolation map can also be chosen $\mathscr{C}^n$.

Another interpolation theorem concern sectors that are introduced later in subsection 4.2.

Definition 1.4.4 (Near translation)**.** For $j = 1, 2$, let Γ_j denote $\mathscr{C}^1$-Jordan arcs parametrized by $t \to z(t)$ which are backward invariant by translation under translation by σ where $Re(\sigma) > 0.$, i.e $\Gamma_j - \sigma \subset \Gamma_j$, and let $h : \Gamma_1 \to \Gamma_2$ be a $\mathscr{C}^1$-diffeomorphism. Then h is said to be a *near translation* or $\mathscr{C}^1$*-bounded* if there exists a constant $C > 1$ such that

$$\forall z \in \Gamma_1,\ |h(z) - z| < C \quad \text{and} \quad \forall t,\ \frac{1}{C} < \frac{|\frac{d}{dt}h(z(t))|}{|\frac{d}{dt}z(t)|} < C \tag{10}$$

Remark. Note that periodic maps are always near translations.

Theorem 1.4.4 (Interpolation on sectors)**.** *Let S_1 and S_2 be two sectors with $\mathscr{C}^2$ outer boundaries. Let $g_{L/R} : \partial_{L/R}S_1 \to \partial_{L/R}S_2$ be analytic maps and $g_{out} : \partial_{out}S_1 \to \partial_{out}S_2$ be a $\mathscr{C}^2$-diffeomorphism. Assume $g_{L/R}$ to be near translations near the sectors vertices when written in the log-linearising coordinates.Then there exists a quasiconformal interpolation map $g : \overline{S_1} \to \overline{S_2}$ equal to the boundary maps on the boundaries.*

2 Notions of Holomorphic Dynamics

The object of this section is to introduce basic knowledge about complex dynamical systems in one variable in the rational and more precisely in the quadratic polynomial case.

The main purpose of holomorphic dynamics is the study of iterated holomorphic mappings from a Riemann surface S to itself. And it requires the introduction of some particular sets characterizing the behaviour of such mappings.

The majority of the proofs can be found in Milnor's lecture book [Mil06] for all the basic properties of Julia and Fatou sets and in Douady and Hubbard Orsay notes [DH09] for the basic properties of polynomial dynamics and the structure of the Mandelbrot set.

2.1 Fatou and Julia sets

Definition 2.1.1 (Normal families)**.** A collection $\mathscr{F}$ of holomorphic maps from a Riemann surface S into another T is called *a normal family* if every infinite sequence of maps of $\mathscr{F}$ contains either a locally uniformly convergent subsequence or a locally uniformly divergent subsequence from T.

As we will consecrate our studies to the case where T is compact equal to $\widehat{\mathbb{C}}$, an easier definition follows.

Definition 2.1.2 (Normal families)**.** A collection $\mathscr{F}$ of holomorphic maps from a Riemann surface S into another compact Riemann surface T is called *a normal family* if its closure $\overline{\mathscr{F}} \subset Hol(S,T)$ is compact, that is every infinite sequence of maps of $\mathscr{F}$ contains a locally uniformly convergent subsequence.

This provides a local dichotomy around every point of S giving the following definitions.

Definition 2.1.3 (Fatou and Julia sets)**.** Let S be a compact Riemann surface and let $f : S \to S$ be a nonconstant holomorphic mapping. We call the *Fatou set* of f its domain of normality of the collection of its iterates $\{f^n\}_{n\in\mathbb{N}}$. We denote it by F_f.

The *Julia set* of f is the complement of its Fatou set, that is $J_f = S \setminus F_f$.

Definition 2.1.4 (Basic definitions)**.** Let S be a Riemann surface, $f : S \to S$ be a conformal map and $z_0 \in S$ be given.

- The *(forward) orbit* of z_0 under f is the sequence $\mathscr{O}(z_0) = \mathscr{O}^+(z_0) = \{f^n(z_0)\}_{n\in\mathbb{N}}$.
- A *critical orbit* is the orbit of a critical point of f.
- If there exists a smallest $p > 0$ such that $f^p(z_0) = z_0$, z_0 is called a *p-periodic point* and its orbit is called a *p-cycle.*
- If there exists a $p, k > 0$ such that $f^{k-1}(z_0) \neq f^{k+p-1}(z_0)$ and $f^k(z_0) = f^{k+p}(z_0)$, z_0 is called a *pre-periodic point.*

Suppose z_0 a p-periodic point and denote its orbit $\mathscr{O}(z_0) = \{z_0, z_1, ..., z_{p-1}\}$.

- Define the *multiplier* of the cycle as $\lambda_{z_0} = (f^p)'(z_0) = f'(z_0).f'(z_1). \dots .f'(z_{p-1})$.

- The dynamics around this cycle depends on the multiplier, that is the orbit is
  - *superattracting* if $|\lambda_{z_0}| = 0$.
  - *attracting* if $0 < |\lambda_{z_0}| < 1$.
  - *repelling* if $|\lambda_{z_0}| > 1$.
  - *neutral* or *indifferent* if $|\lambda_{z_0}| = 1$.

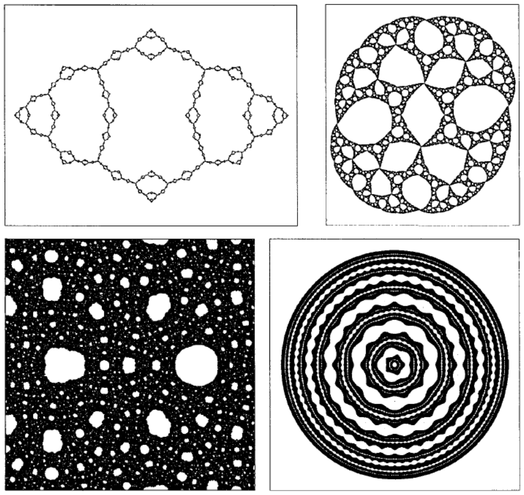
  And by writing $\lambda_{z_0} = e^{i2\pi\alpha}$, $\alpha \in \mathbb{R}/\mathbb{Z}$, the last case splits into:
  - *parabolic* or *rationally indifferent* if $\alpha \in \mathbb{Q}$.
  - *irrationally indifferent* if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.
- The *basin of attraction* of an attracting or a parabolic p-cycle $\mathscr{O}(z_0)$ is defined as the set $\mathscr{A}_f(z_0) = \{z \in S \,|\, f^{np}(z) \to z_i \, for \, z_i \in \mathscr{O}(z_0)\}$.
- The *immediate basin of attraction* $\mathscr{A}_0(z_0)$ of a fixed point z_0 is the connected component of its basin of attraction containing z_0.
- The *immediate basin of attraction* $\mathscr{A}_0(z_0)$ of an attracting or a parabolic p-cycle $\mathscr{O}(z_0)$ is the union of the immediate basin of attractions of the elements of its cycles seen as fixed points of f^p.

Proposition 2.1.1 (Properties of Julia and Fatou sets). *Let S be a simply connected Riemann surface and let $f : S \to S$ be a rational map. Then we have the following properties:*

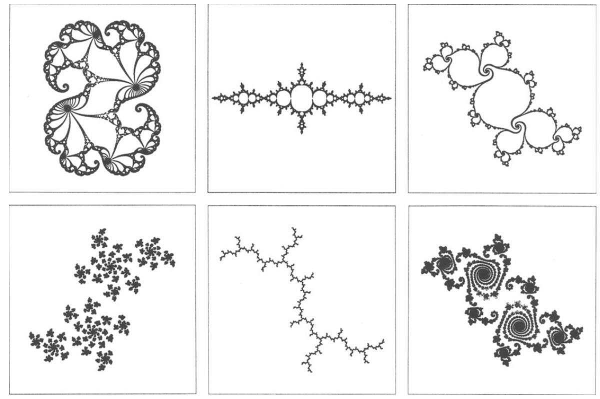
1. *For all $k > 0$, $J_f = J_{f^k}$.*
2. *$f(J_f) = J_f = f^{-1}(J_f)$.*
3. *Every attracting cycle and its basin of attraction belongs to F_f.*
4. *If $\mathscr{A}$ is the basin of attraction of a an attracting cycle, then $\partial\mathscr{A} = J_f$.*
5. *All repelling and parabolic points belong to J_f.*
6. *$J_f \neq \emptyset$ and is uncountable.*
7. *J_f has either a fixed repelling point or a parabolic point of multiplier equal to 1.*
8. *J_f has either an empty interior or is equal to S.*
9. *J_f has no isolated point.*
10. *J_f is either connected or has uncountably many connected components.*
11. *Repelling periodic points of f are dense in J_f.*

All of these properties give a fractal aspect to Julia sets as shown in the examples of figure 3.

As our later interest will be focused on quadratic polynomial, we study their particular case in the next subsection.



(a) Examples for rational maps



(b) Example for quadratic polynomials

Figure 2: Examples of Julia sets, from [Mil06]

2.2 Polynomial dynamics

In the polynomial case, the infinite point of $\widehat{\mathbb{C}}$ is a superattracting fixed point of f and it has a basin of attraction.

Definition 2.2.1 (Filled Julia set)**.** Let f be a polynomial map. The *filled Julia set* K_f of f is the complementary of the basin of infinity $\mathscr{A}(\infty)$, that is the set of complex numbers having a bounded orbit.

And by using the maximum modulus principle for a polynomial map, we obtain some immediate properties of such a set.

Proposition 2.2.1. *Let f be a polynomial of degree at least 2. The filled Julia set of f is compact in $\mathbb{C}$ with a connected complement and such that $\partial K_f = J_f$. Its interior is the union of all bounded component of the Fatou set F_f which are simply connected.*

Theorem 2.2.1 (Connectedness of K_f)**.** *Let f be a polynomial of degree $d \geq 2$.*

- *If the filled Julia set of f contains all of the finite critical orbits, K_f and J_f are connected. In this case, $\mathscr{A}(\infty)$ is conformally isomorphic to $\mathbb{C} \setminus \overline{\mathbb{D}}$ under an isomorphism $\phi : \mathbb{C} \setminus K_f \to \mathbb{C} \setminus \overline{\mathbb{D}}$ called the Böttcher map conjugating f to the d-th power map $w \to w^d$.*

- *If $\mathscr{A}(\infty)$ contains at least one critical orbit, both K_f and J_f have uncountably many connected components.*

- *If $\mathscr{A}(\infty)$ contains all of the critical orbits, $K_f = J_f$ is a Cantor set.*

Remark. In the particular case of quadratic polynomials, since it has only one critical point, the Julia set is either connected or a Cantor set.

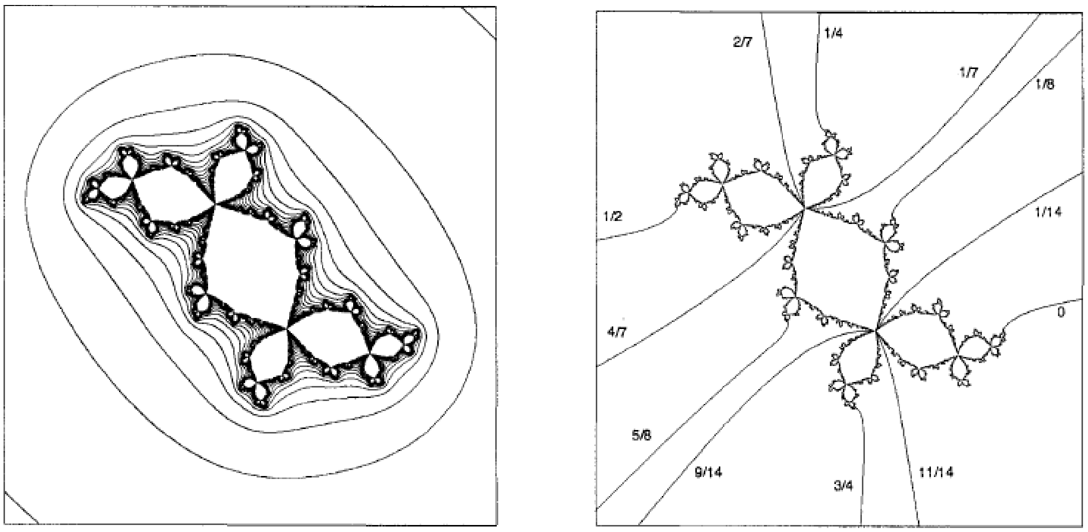
Definition 2.2.2 (Equipotentials and external rays)**.** Let f be a polynomial map with a connected Julia set, and let ϕ be its Böttcher's map. We call the *Green's function* for K_f the continuous function $G : \mathbb{C} \to \mathbb{R}$ defined by

$$G(z) = \begin{cases} log|\phi(z)| > 0 & if z \in \mathbb{C} \setminus K_f \\ 0 & if z \in K_f \end{cases} \tag{11}$$

Note that

$$G(f(z)) = dG(z) \tag{12}$$

- for all $c > 0$, $G^{-1}(c)$ is called an *c-equipotential curve* around K_f. And f maps the c-equipotential to the dc-equipotential by a q-to-one covering.
- for all $\theta \in \mathbb{R}/\mathbb{Z}$, the set $R_\theta = \{z \in \mathbb{C} \setminus K_f \,|\, arg(\phi(z)) = 2\pi\theta\}$ is called the *external ray* of angle θ. In this case, θ is called the *external angle* of the external ray. And f maps R_θ to $R_{d\theta}$.



(a) Equipotentials (b) External rays

Figure 3: The Douady rabbit $z \to z^2 + c$ with $c \approx -0.12256 + 0.74486i$, from [Mil06]

If the angle θ if (pre-)periodic under multiplication by d, the ray R_θ is *(pre-)periodic.*

Let $\gamma(\theta) = \lim_{r \to 1} \phi^{-1}(re^{i2\pi\theta})$. Whenever this limit exists, we say that the ray R_θ *lands* at $\gamma(\theta) \in J_f$.

Proposition 2.2.2. *With the previous notations,*

1. *For all $\theta \in \mathbb{R}/\mathbb{Z}$ outside of a zero measure set, R_θ has a well defined landing point $\gamma(\theta) \in J_f$. And for each $z \in J_f$, $\gamma^{-1}(z)$ is of measure zero.*
2. *We have the equivalence: Every external ray lands and γ is continuous iff J_f is locally connected iff K_f is locally connected. In this case, $\gamma : \mathbb{T} \to J_f$ is called the Caratheodory semiconjugacy for J_f.*
3. *If a periodic ray lands at a point z, then only finitely many rays land at this point z and all these rays are periodic of the same period (which period may be greater than z's).*
4. *Every periodic external ray lands at periodic point of f which is either repelling or parabolic.*
5. *Every pre-periodic external ray lands at a pre-periodic point of f.*
6. *Every repelling or parabolic periodic point is the landing point of at least one periodic ray.*
7. *Every rational external ray lands.*

2.3 The Mandelbrot set

In this subsection, we will concentrate a more particular case of the previous subsection which is the case of quadratic polynomials, and more precisely, the connectedness locus of such maps.

We consider the family $\{Q_c : z \to z^2 + c\}_{c\in\mathbb{C}}$ which represents every quadratic polynomial up to affine conjugacy.

Notation. We denote by F_c, J_c and K_c respectively the Fatou set, the Julia set and the filled Julia set of Q_c.
We also denote by $R_c(\theta)$ the external ray for Q_c of angle θ.

Definition 2.3.1 (The Mandelbrot set)**.** The *Mandelbrot set* $\mathscr{M}$ is the connectedness locus of the family of quadratic polynomials $\{Q_c : z \to z^2 + c\}_{c\in\mathbb{C}}$.

Remark. By using theorem 2.2.1, we get the dichotomy:

- If $0 \in K_c$, $c \in \mathscr{M}$.
- If $0 \notin K_c$, $c \notin \mathscr{M}$.

Hubbard and Douady proved in [DH85] and [DH09] the next theorem.

Theorem 2.3.1. *ϕ_c extended to c which defines the map*

$$\begin{aligned}\Phi : \mathbb{C}\setminus\mathscr{M} &\to \mathbb{C}\setminus\overline{\mathbb{D}}\\ c &\to \phi_c(c)\end{aligned}$$

This mapping is conformal.

Corollary 2.3.1.1. *The Mandelbrot set $\mathscr{M}$ is connected. Furthermore, it is simply connected.*

Definition 2.3.2. In the same way as Julia sets, Φ defines a Green's function $G_M = log\,|\Phi|$, equipotential curves and external rays for $\mathscr{M}$.

Definition 2.3.3. We set $\mathscr{M}' = \{c \in \mathbb{C}\,|\,Q_c \text{ has an attracting cycle}\} \subset \mathring{\mathscr{M}}$.

The connected components of $\mathscr{M}'$ are called the *hyperbolic components* of $\mathscr{M}$. Each component corresponds to a unique periodicity of the attracting cycles, and it is the unique attracting cycle for each element Q_c.

Theorem 2.3.2. *Let W be a hyperbolic component of $\mathscr{M}$. Let the application giving the multiplicity of each cycle be denoted by $\rho_W : W \to \mathbb{D}$. Then, ρ_W is a conformal isomorphism that extends continuously from the boundary of W to $\mathbb{S}^1$.*

Remark. The proof of the theorem is one of the first examples of quasiconformal surgery which fits perfectly in this report. But I prefer to omit this proof which is done in a more general setting in section 4.1 of [BF14].

Definition 2.3.4. For every hyperbolic component W of $\mathscr{M}$.

- $\rho_W^{-1}(0)$ is called the *center* of W.
- $\rho_W^{-1}(1)$ is called the *root* of W.
- for each $c \in W$, $arg(\rho_W(c))$ is called the *internal angle* of c.

And we define the following subsets of $\mathscr{M}$

- $\mathscr{D}_0 = \{$ centers of hyperbolic components $\} = \{c \in \mathbb{C}\,|\,0$ is a (superattracting) periodic point of $Q_c\}$.
- $\mathscr{D}_1 = \{$ roots of hyperbolic components $\} = \{c \in \mathbb{C}\,|$ the cycle is parabolic$\}$.
- $\mathscr{D}_2 = \{$ Misiurewicz points$\} = \{c \in \mathbb{C}\,|\,0$ is strictly preperiodic$\}$.

Proposition 2.3.1. $\partial\mathscr{M} \subset \overline{\mathscr{D}_i}$, $\forall i \in \{0, 1, 2\}$.

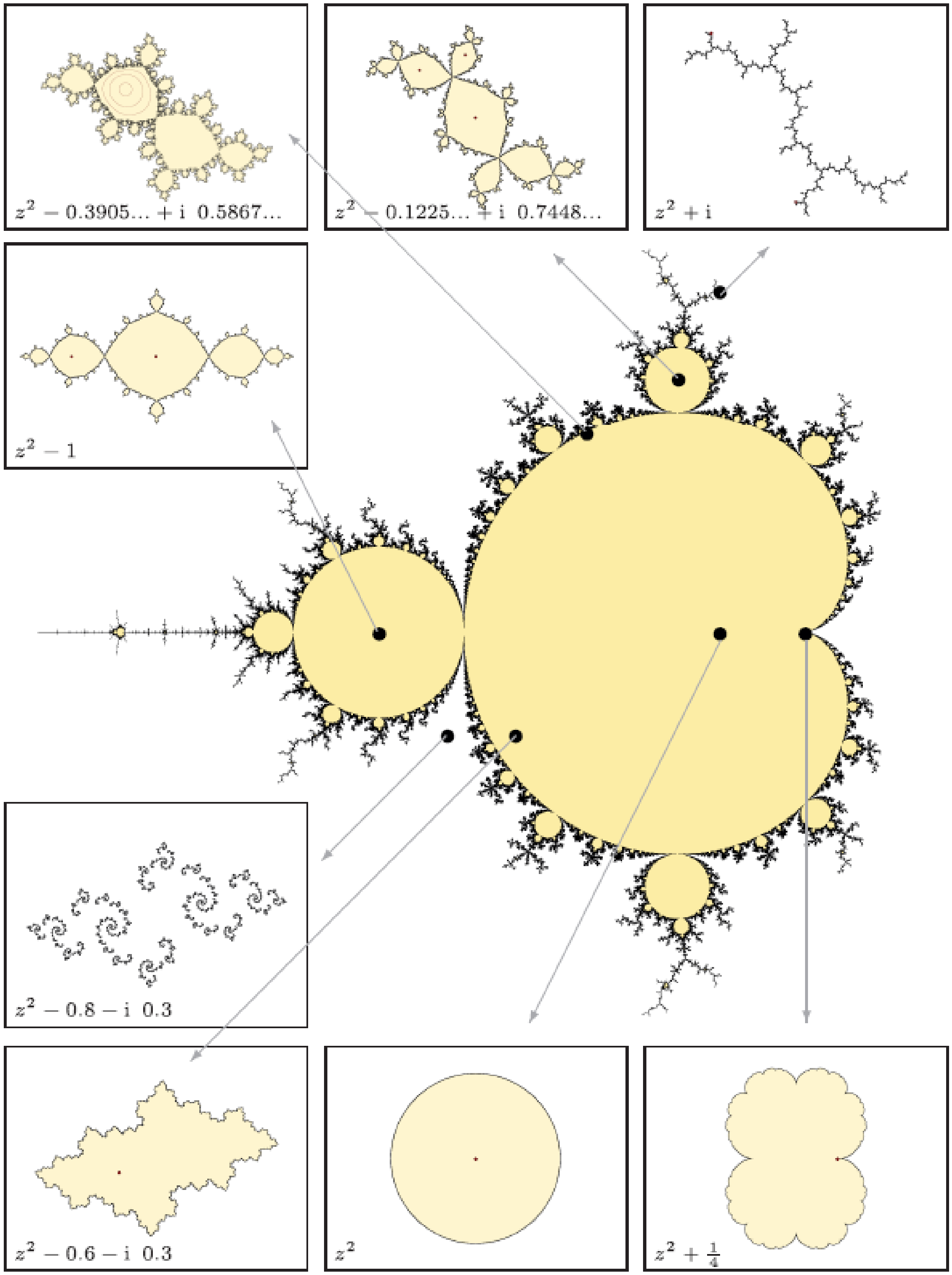


Figure 4: The Mandelbrot set, from [BF14]

2.4 Polynomial like dynamics

This subsection uses the quasiconformal tools introduced in section 1. And it shows the importance of polynomial dynamics that do appear locally in dynamics generated by a much greater family of holomorphic maps.

Definition 2.4.1 (Polynomial-like mappings)**.** Let $U, V \subset \mathbb{C}$ be bounded simply connected domains with analytic boundaries and such that $\overline{U} \subset V$. The triplet (f, U, V) is called a *polynomial-like mapping of degree d* if $f : U \to V$ is holomorphic and proper of degree d.

Definition 2.4.2 (Julia set and Filled Julia set)**.** Let (f, U, V) be a polynomial-like mapping. The *filled Julia set* of f is defined as $K_f = \bigcap_{n>0} f^{-n}(V)$, and its *Julia set* is defined as $J_f = \partial K_f$.

Before stating the main theorem of this subsection relating polynomial-like mappings to polynomial mappings, we need to indtoduce a new king of conjugacy.

Definition 2.4.3 (Hybrid equivalence)**.** Two polynomials f and g are *hybrid equivalent* if there exist neighbourhoods U_f and U_g of K_f and K_g respectively, and a quasiconformal conjugacy $\phi : U_f \to U_g$ between f and g satisfying $\overline{\partial}\phi = 0$ almost everywhere on K_f. We write $f \sim_{hyb} g$.

Remark. By Weyl's lemma (Theorem 1.2.3), the condition satisfied by ϕ means that it is holomorphic in $\mathring{K}_f$.

This type of conjugacy is the strongest type of conjugacy when Julia sets are connected as shown in Prop.21 of [DH85].

Theorem 2.4.1. *Let $f, g \in Pol_d$ with connected Julia sets. If f, and g are hybrid equivalent, they are affine conjugate.*

But this is completely false for polynomials with disconnected Julia sets as the first part of the following proposition states it.

Proposition 2.4.1 (Conjugacy classes of Q_c)**.** *(i) Any two polynomials of $\mathbb{C} \setminus \mathscr{M}$ are hybrid equivalent, but no different polynomials of $\mathscr{M}$ are.*

(ii) All polynomials in a given hyperbolic component with the exception of its centre are globally quasiconformally conjugate. If we only require the conjugacies to hold in a neighbourhood of the Julia set, then the centre is also included.

(iii) elements of $\partial\mathscr{M}$ are the unique representatives of their quasiconformal conjugacy classes, that is if $c \in \mathscr{M}$ and $c' \in \mathbb{C}$ such that Q_c and $Q_{c'}$ are quasiconformally conjugate, then $c = c'$.

Remark. The second point of the proposition appears in the proof of the theorem 2.3.2 while changing the multiplier inside a hyperbolic component holomorphically using the parametrized version of the integrability theorem 1.3.2.

We now state and prove the main theorem of this subsection.

Theorem 2.4.2 (The straightening theorem)**.** *Every polynomial-like mapping (f, U, V) of degree d is hybrid equivalent to a polynomial P of degree d. Moreover, if K_f is connected, P is unique up to affine conjugation.*

Proof. This proof will be the introduction to quasiconformal surgery in this report.
The idea of the proof is to glue f in U to the map $z \to z^d$ in $\mathbb{C} \setminus V$ via a well chosen quasiconformal ψ gluing map conjugating them.

Construction of ψ. Let $r > 0$ and $\mathscr{R} : \mathbb{C} \setminus \overline{V} \to \mathbb{C} \setminus \overline{\mathbb{D}_{r^d}}$ fixing ∞ and extended continuously by an analytic map $\phi_1 : \partial V \to \mathbb{S}^1_{r^d}$ using theorem 1.4.2.

Since $f : U \to V$ is proper of degree d, and U and V have analytic boundaries, f extends continuously to an orientation preserving d-to-one map $f : \partial U \to \partial V$. Take $\psi_2 : \partial U \to \mathbb{S}^1_r$ be a lift of ψ_1 such that the following diagram commutes.

$$\begin{array}{ccc} \partial U & \xrightarrow{\psi_2} & \mathbb{S}^1_r \\ \downarrow f & & \downarrow z \to z^d \\ \partial V & \xrightarrow{\psi_1} & \mathbb{S}^1_{r^d} \end{array}$$

Let $A_0 = V \setminus U$. A_0 is a quasiannulus and ψ_1 and ψ_2 are its analytic boundary maps. By theoreom 1.4.1, these maps have a quasiconformal interpolation $\psi : \overline{A_0} \to \overline{\mathbb{A}_{r,r^d}}$.

Gluing. We construct a new map $F : \mathbb{C} \to \mathbb{C}$ by gluing as mentioned in the beginning of the proof. Define F in $\mathbb{C}$ by

$$F(z) = \begin{cases} f(z) & \text{if } z \in U \\ \mathscr{R}^{-1}(\psi(z)^d) & \text{if } z \in V \setminus U \\ \mathscr{R}^{-1}(\mathscr{R}(z)^d) & \text{if } z \in \mathbb{C} \setminus V \end{cases}$$

F is clearly a quasiregular map as it is continuous by definition of the boundary maps of ψ and it is the composition of holomorphic mappings with quasiconformal mappings.

Changing the complex structure. The set A_0 is a fundamental domain for F, that is every orbit intersect it at most once. Hence, if we define for all $n \geq 0$ $A_n = F^{-n}(A_0) = f^{-n}(A_0)$, we obtain pairwise disjoint sets.
As F is quasiregular, by proposition 1.2.2, we can consider the pullbacks of almost complex structures using F. Thus, we can define the almost complex structure

$$\sigma(z) = \begin{cases} \psi^*\sigma_0(z) & \text{if } z \in A_0 \\ (F^n)^*\sigma_0(z) & \text{if } z \in A_n \\ \sigma_0(z) & \text{elsewhere} \end{cases}$$

We obtain by construction an F-invariant complex structure.

Finding the conformal mapping. Since only all but a finite number of pullbacks are done using the conformal map f, the dilatation of σ is finite, and more precisely, it is equal to that of $\psi^*\sigma_0$. Thus, the dilatation of σ is bounded and we can apply the integrability theorem 1.3.1. Hence, there exists a quasiconformal map $\phi : \mathbb{C} \to \mathbb{C}$ such that $\phi^*\sigma_0 = \sigma$.

Consider $P = \phi^{-1} \circ F \circ \phi$ which gives the flowing commutative diagram:

$$\begin{array}{ccc} (\mathbb{C}, \sigma_0) & \xrightarrow{P} & (\mathbb{C}, \sigma_0) \\ \downarrow \phi & & \downarrow \phi \\ (\mathbb{C}, \sigma) & \xrightarrow{F} & (\mathbb{C}, \sigma) \end{array}$$

We see that $P * \sigma_0 = \sigma_0$ on all of $\mathbb{C}$. By Weyl's lemma 1.2.3, P is holomorphic of degree d, hence it is a polynomial.

Note that in K_f, $\sigma = \sigma_0$ which implies that $\overline{\partial}\phi = 0$ on K_f, meaning that f is hybrid equivalent to the polynomial P of degree d, which concludes the proof. □

Remark. The procedure used in the proof is called *quasiconformal surgery* which is a general way to find to find conformal mappings using quasiconformal mappings constructed through topological gluing.

The proof could have been proven using more general theorems of quasiconformal surgery like the *Shishikura principles* or *Sullivan's straightening theorem* stated and proven in the fifth chapter of [BF14].

3 Straightening maps

3.1 Copies of the Mandelbrot set

In this subsection, we will properly define copies of the Mandelbrot set that can appear in different dynamical systems. And because the Mandelbrot set appears in parameter spaces and not phase spaces, we must consider some particular families of conformal mappings.

Definition 3.1.1 (Holomorphic family)**.** Let Λ be a complex analytic manifold and $\mathscr{F} = (f_\lambda, U_\lambda, V_\lambda)_{\lambda\in\Lambda}$ a family of polynomial-like mappings. Set $\mathscr{U} = \{(\lambda, z) \,|\, z \in U_\lambda\}$, $\mathscr{V} = \{(\lambda, z) \,|\, z \in V_\lambda\}$ and $f(\lambda, z) := (\lambda, f_\lambda(z))$.

$\mathscr{F}$ is a *holomorphic family* of polynomial-like mappings if it satisfies the following properties:

(i) $\mathscr{U}$ and $\mathscr{V}$ are homeomorphic over Λ to $\Lambda \times \mathbb{D}$.

(ii) The projection from the closure of $\mathscr{U}$ in $\mathscr{V}$ to Λ is proper.

(iii) $f : \mathscr{U} \to \mathscr{V}$ is holomorphic and proper.

In this case, all polynomial-like mappings have the same degree called *degree of the family* $\mathscr{F}$.
We define the *connectedness locus* of $\mathscr{F}$ as $\mathscr{C}_{\mathscr{F}} = \{\lambda \in \Lambda \,|\, K_{f_\lambda} \text{ is connected }\}$.

Definition 3.1.2 (Straightening map)**.** Let $\mathscr{F} = (f_\lambda, U_\lambda, V_\lambda)_{\lambda\in\Lambda}$ be a holomorphic family of polynomial-like mappings of degree 2. We denote its connectedness locus by $\mathscr{M}_{\mathscr{F}}$.
By the straightening theorem 2.4.2, for each $\lambda \in \mathscr{M}_{\mathscr{F}}$, there exists a unique complex number $c \in \mathscr{M}$ such that $f_\lambda \sim_{hyb} Q_c$. Hence the map

$$\begin{aligned} \chi : \mathscr{M}_{\mathscr{F}} &\longrightarrow \mathscr{M} \\ \lambda &\longmapsto c \end{aligned}$$

is well defined and is called the *straightening map* of the family $\mathscr{F}$.

Compactness and bijectivity of straightening maps are deeply studied even in higher dimension in [IK12].

Theorem 3.1.1 (Dependence on parameters)**.** *Let $\mathscr{F} = (f_\lambda, U_\lambda, V_\lambda)_{\lambda\in\Lambda}$ be a holomorphic family of quadratic-like mappings.*

(i) The straightening map χ extends to a continuous map $\chi : \Lambda \to \mathbb{C}$ such that for all $\lambda \in \Lambda$, $f_\lambda \sim_{hyb} Q_{\chi(\lambda)}$.

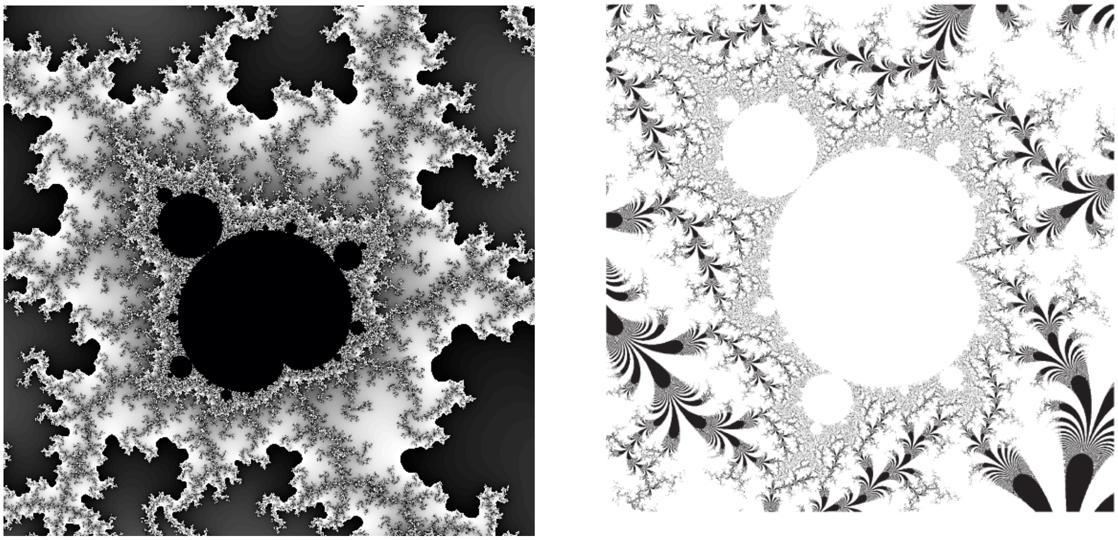
(ii) Suppose Λ homeomorphic to $\mathbb{D}$. If $\chi : \mathscr{M}_{\mathscr{F}} \to \chi(\mathscr{F}) \subset \mathscr{M}$ is proper and $\chi(\mathscr{F})$ is connected, then χ is a ramified covering. Moreover, if $\mathscr{M}_{\mathscr{F}} \subset\subset \Lambda$, then $\chi : \mathscr{M}_{\mathscr{F}} \to \mathscr{M}$ is a ramified covering of finite degree d equal to the number of rotations done by $f_\lambda - \omega_\lambda$ (where ω_λ is the critical point of f_λ) around zero when λ turns once around $\mathscr{M}_{\mathscr{F}}$.

See Chapter II of [DH85] for a proof.

Definition 3.1.3 (Mandelbrot-like families)**.** With the notations of the last theorem, we have the two following definitions:

- Under the hypothesis of (ii), if $\mathscr{M}_{\mathscr{F}} \subset\subset \Lambda$ and if $d = 1$, $\chi : \mathscr{M}_{\mathscr{F}} \to \mathscr{M}$ is a homeomorphism and in this case, $\mathscr{F}$ is called a *Mandelbrot-like family.*

- If $\chi : \mathscr{M}_{\mathscr{F}} \to \mathscr{M} \setminus \{\frac{1}{4}\}$ is a homeomorphism, $\mathscr{F}$ is called a *semi-Mandelbrot-like family.*



(a) A small copy of $\mathscr{M}$ inside itself.

(b) Copy of $\mathscr{M}$ in the parameter plane of an entire transcendental family of maps.

Figure 5: Copies of the Mandelbrot set

Definition 3.1.4 (Primitive hyperbolic component)**.** Let $c_0 \in \mathscr{D}_0$ be a center of a hyperbolic component W of $\mathscr{M}$, and let $c_1 \in \mathscr{D}_1$ be its root. Denote by $\mathscr{A}_0(c_1)$ the immediate basin of attraction of the parabolic point p of Q_{c_1}. We say that W is *primitive* if the flower of p has only one attracting petal, that is $\mathscr{A}_0(c_1) \setminus \{p\}$ has only one connected component.

Theorem 3.1.2 (Modulation theorem)**.** *Let $c_0 \in \mathscr{D}_0$ be a center of a hyperbolic component W. Only two scenarios can happen:*

1. *W is a primitive component of $\mathscr{M}$ of period k, and in this case, we can find a neighbourhood Λ of $\overline{W}$ and two families of open sets $(U_c, V_c)_{c\in\Lambda}$ of $\mathbb{C}$ such that $\{f_c = Q_c^k : U_c \to V_c\}_{c\in\Lambda}$ is a Mandelbrot-like family.*

2. *W is not a primitive component of period k, and in this case, we can construct Λ, U_c and V_c such that Λ is a neighbourhood of $\overline{W} \setminus \{c_1\}$ where c_1 is the root of W, and $\{f_c = Q_c^k : U_c \to V_c\}_{c\in\Lambda}$ is a semi-Mandelbrot-like family.*

This theorem was already known by A.Douady and J.H.Hubbard, but a properly written proof figures in the first part of a paper by P.Haïssinsky [Haï00].

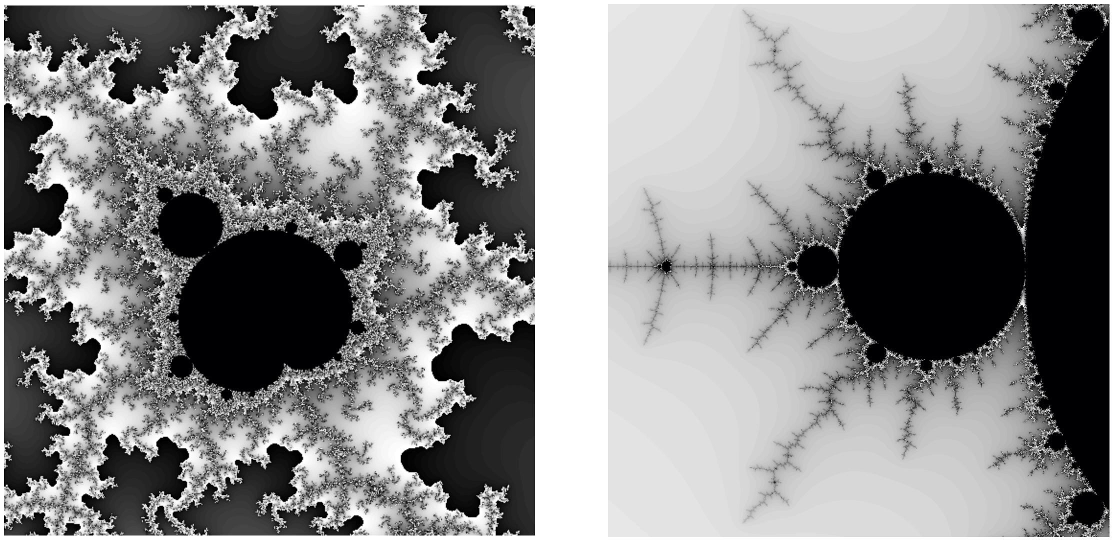
Remark. M.Lyubich proved that the copies of the Mandelbrot set around a primitive component are quasiconformal to $\mathscr{M}$ itself.

Notation. We denote by $\mathscr{M}_{c_0}$ the copy of the Mandelbrot around W, i.e $\mathscr{M}_{c_0} = \chi_{c_0}^{-}1(\mathscr{M})$, where χ_{c_0} is the straightening map of the constructed (semi-)Mandelbrot-like family.

3.2 Action on the Julia sets

These small copies of the Mandelbrot have their Julia sets distributed in some elegant manner which description is the object of this subsection.

A useful way to do so is to introduce the fundamental notion of renormalization which provides an alternative way to construct straightening maps and an effective way to study them. I will define the notion in the quadratic case which interests us, but for further knowledge see [McM94].



(a) Primitive component (b) Non primitive component

Figure 6: Small copies of the Mandelbrot sets

Definition 3.2.1 (Renormalization)**.** A quadratic polynomial P with K_P connected is said to be *renormalizable* if there is an open subset $V \subset \mathbb{C}$ containing 0 and an integer $k > 1$ such that one component U of $P^{-k}(V)$ relatively compact in V and the restriction $P_V = P^k_{|U} : U \to V$ is a quadratic-like mapping with its own filled Julia set K_{P_V} connected.

Through the proof of the proposition 4.2 of [Haï00] which is nothing but an adaptation of the proof of the theorem 5.7 b) of [Hub93] , we get the following theorem.

Theorem 3.2.1 (Straightening map)**.** *Let W be a hyperbolic component of $\mathscr{M}$ of period $k > 0$. Let $\widehat{\mathscr{D}_2}$ be the set of landing points of dyadic external rays of $\mathscr{M}$ on $\mathscr{M}_{c_0}$. Then for every $c \in \mathscr{M}_{c_0} \setminus \widehat{\mathscr{D}_2}$, Q_c is k-renormalizable with renormalization map hybrid equivalent to $Q_{\chi_{c_0}(c)}$.*

Remark. This result shows that the Julia set of K_c for $c \in \mathscr{M}_{c_0}$ resembles $K_{\chi_{c_0}(c)}$. And by taking images and preimages, we get that these copies are dense in K_c.

The exact shape of K_c has been determined by Haïssinsky in the second part about modulation operator of [Haï00]. We state the main result of the paper after introducing a topological operation.

Definition 3.2.2 (Modulation of compact sets)**.** Let K be a locally and simply connected compact such that $\mathring{K}$ has a countably many connected components $\pi_0(\mathring{K}) = \{U_i\}_{i\in\mathbb{N}}$ which are Jordan domains with associated Riemann mappings $\phi_{U_i} : \mathbb{D} \to U_i$ that extends to the boundaries by $\gamma_U : \mathbb{S}^1 \to \partial U_i$, and such that their pairwise intersection contains at least one point so that it is possible to define a projection $\pi_{U_i} : K \to \overline{U_i}$ fixing U_i and sending every point z of $K \setminus U_i$ to the unique point of ∂U_i which can be connected to z by a path outside of U_i (it is locally constant on $K \setminus \overline{U_i}$).

Let $\sim$ be the equivalence relation on K defined by

$$x \sim y \quad \leftrightarrow \quad x = y \text{ or } \exists U \in \pi_0(\mathring{K});\, x, y \in \overline{U}$$

Let $g : K \to \widehat{K}$ be the projection map.

Now, let $L \subset \mathbb{C}$ be a simply connected compact of the complex plane with an a.e well defined Caratheodory curve $\gamma_L : \mathbb{T} \to \partial L$. Set L_i copies of L.

For every $U \in \pi_0(\mathring{K})$, define $\psi_U = \gamma_L \circ \gamma_U^{-1} : \pi(K \setminus \overline{U}) \to \partial L$.

The set K *modulated by* L is the compact set

$$K_L = \left\{ (\xi, \xi_i) \in \widehat{K} \times \prod_i L_i \,|\, \xi_i = \begin{cases} \psi_i \pi_i(g^{-1}(\xi)) & \textit{if } g(U_i) \neq \xi_i \\ \psi_i \pi_i(g^{-1}(U_i)) & \textit{otherwise, i.e } \exists k \neq i \textit{ s.t } g(U_k) = \xi \end{cases} \right\}.$$

Interpretation. In a less complicated manner, K_L is the set obtained by taking K and replacing all the connected components of its interior by copies of L.

Theorem 3.2.2 (Shape of the Julia sets). *Let $c_0 \in \mathscr{D}_0 \setminus \{0\}$ and $c \in \mathscr{M}$. Let $\widetilde{c} = \chi_{c_0}^{-1}(c)$. Then $K_{\widetilde{c}}$ is homeomorphic to $K_{c_0 K_c}$.*

See figure 7.

4 Inverse construction

With the notations of theorem 7, knowing that the shape of $K_{\widetilde{c}}$ is a gluing between K_c and K_{c_0}, one can ask the question of if it is possible to construct $\widetilde{c} = \chi_{c_0}^{-1}(c)$ by quasiconformal surgery.

Let $c_0 \in \mathscr{D}_0 \setminus \{0\}$ center of a component of period $k > 0$ and let $c \in \mathscr{M}$.

4.1 c_0's phase plane

We start by dividing c_0's phase plane into well-selected domains according to the dynamics of Q_{c_0}.

Let V_0 be the connected component of $\mathring{K}_{c_0}$ containing 0, which is a topological circle. And let $V_i = Q_{c_0}^i(V_0)$ for all $i = 1, .., k-1$. Let W_j be the connected components of $\mathbb{C} \setminus \bigcup_i \overline{V_i}$.

By renormalization, $Q_{c_0}^k$ is hybrid equivalent to $z \to z^2$. Let x_0 and $y_0 \in \partial V_0 \subset J_{c_0}$ be the points of $\mathbb{C}$ equivalent respectively to 1 and -1. Note that $Q_{c_0}^k(y_0) = x_0$.

Since x_0 is a repelling periodic point Q_{c_0} it is the landing point of periodic external rays by property 6 of proposition 2.2.2. As $Q_{c_0}^k(y_0) = x_0$, y_0 is also the landing point of pre-periodic external rays. Let $R(x_0)$ and $R'(x_0)$ be the two external rays landing at x_0 and $R(y_0)$ and $R'(y_0)$ be the two external rays landing at y_0 such that by setting $\ell(z) = R(z) \cup \{z\} \cup R'(z)$ for $z = x_0, y_0$, we have the part of the plane between $\ell(x_0)$ and $\ell(y_0)$ contains no other external ray landing at x_0 or y_0.

For $i = 1, .., k-1$, for define in the same way x_i, y_i, $R(z)$, $R'(z)$ and $\ell(z)$ for $z = x_i, y_i$.

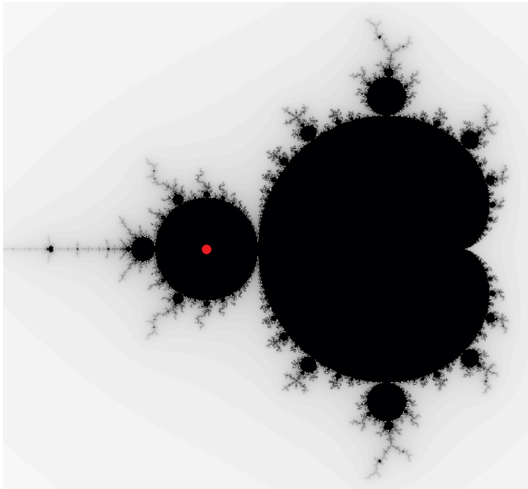
Proposition 4.1.1. *We get the following properties:*

1. *All V_i are topological discs and Q_{c_0} has the following dynamics:*

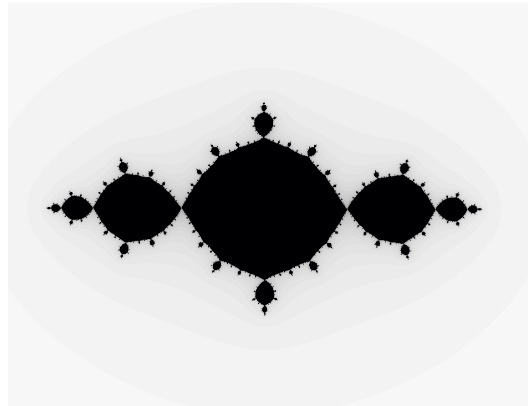
$$V_0 \xrightarrow{2-to-1} V_1 \xrightarrow{1-to-1} V_2 \xrightarrow{1-to-1} \cdots \xrightarrow{1-to-1} V_{k-1} \xrightarrow{1-to-1} V_0$$

2. *For all $i = 1, .., k-1$, $x_i = Q_{c_0}(x_{i-1})$.*
3. *$Q_{c_0}(y_0) = x_1$ and $\forall i = 2, .., k-1$, $y_i = Q_{c_0}(y_{i-1})$.*
4. *For all $i = 1, .., k-1$, $\ell(x_i) = Q_{c_0}(\ell(x_{i-1}))$.*
5. *$Q_{c_0}(\ell(y_0)) = x_1$ and $\forall i = 2, .., k-1$, $\ell(y_i) = Q_{c_0}(\ell(y_{i-1}))$.*

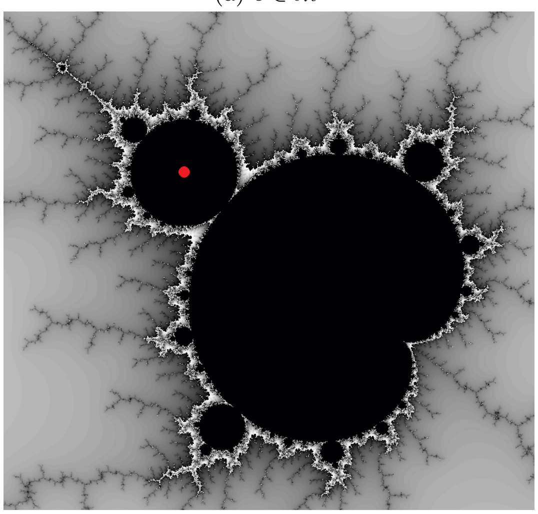
Proof. 1. 0 is the only critical point of Q_{c_0}, thus since V_0 is a neighbourhood of 0, $Q_{c_0} : V_0 \to V_1$ is a ramified covering of degree 2, and with ramification point located at the origin. The other sets do not contain 0 and $Q_{c_0} : V_{k-1} \to V_0$ is a covering with V_0 a Jordan domain. Then V_{k-1} is a Jordan domain and $Q_{c_0} : V_{k-1} \to V_0$ is an isomorphism. An iteration gives the result.



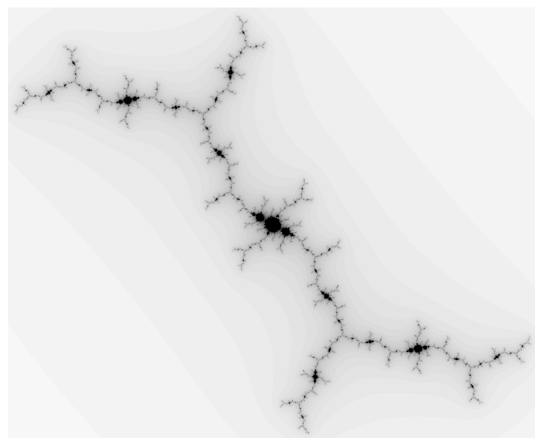
(a) $c \in \mathscr{M}$



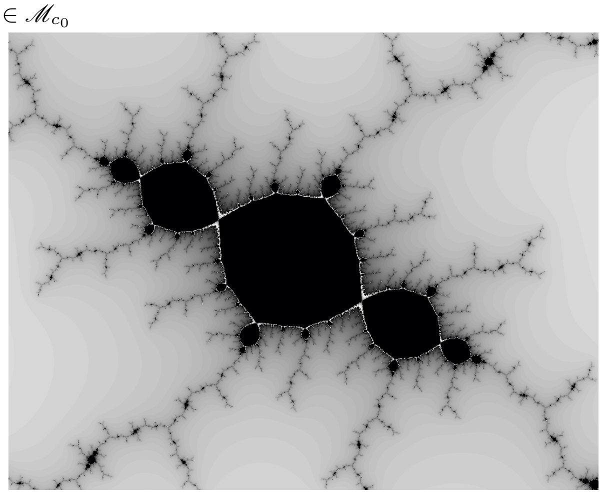
(b) The Julia set K_c



(c) $\widetilde{c} \in \mathscr{M}_{c_0}$



(d) The Julia set $K_{\widetilde{c}}$



(e) Zoom in $K_{\widetilde{c}}$ around the origin

Figure 7: Shapes of Julia sets inside a small copy $\mathscr{M}_{c_0}$ of $\mathscr{M}$

2. For $i = 1,..,k-1$, $Q^i_{c_0}(x_0) \in \partial V_i$ is a k-periodic point and hence a fixed point for the normalized polynomial around V_i given by $z \to z^2$ which has a unique fixed point in $\mathbb{S}^1$, corresponding to x_i.

3. With the same method, by considering the non fixed preimage of the fixed point of $Q^k_{c_0}$ which is the equivalent to -1, we get the result for all $i = 2,..,k-1$. Now, we have $Q_{c_0}(y_0) = Q^{-1}_{c_0|V_{k-1}} \circ Q^{-1}_{c_0|V_{k-2}} \circ \cdots \circ Q^{-1}_{c_0|V_1} \circ Q^k_{c_0}(y_0) = x_1$.

4. and 5. are true since conformal maps preserve orientation and hence the order of external rays around a point.

□

Now consider G_{c_0} the Green's function of Q_{c_0} and $\eta > 0$ a constant to be chosen later. Let $D'_{c_0} = \{G_{c_0} < \eta\}$ and $D_{c_0} = \{G_{c_0} < 2\eta\}$.

4.2 c's phase plane

We divide the c's phase plane using invariant sectors.

Definition 4.2.1 (Sectors)**.** Let P be a polynomial of degree d and let $\zeta \in J_P$ be a periodic point of period $q > 0$. We define a *bounded sector* S with vertex ζ a simply connected bounded domain satisfying $S \subset P^q(S)$ bounded by two arcs γ_R and γ_L starting atζ such that $\gamma_j \subset P^q(\gamma_j)$ and a third arc connected the two corners of γ_j not equal to ζ. The notations are such that γ_R is at the right of S and γ_L at its left. We denote $S = \backslash\gamma_L, \gamma_R/$.
Considering the quotient S/P^q we get a fundamental domain equivalent to an annulus A_S. Define the *opening modulus* of S the modulus of A_S, i.e $mod_\zeta(S, P^q) = mod(A_S)$.

Let $\phi_P : \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C} \setminus K_P$ be the inverse of the Böttcher coordinate of P. Let $\mathbb{H}_r$ be the righta hald-plane of $\mathbb{C}$, and $M_d : z \to dz$ be the multiplication map by d. $\phi_P \circ exp : \mathbb{H}_r \to \mathbb{C} \setminus K_P$ conjugates P to M_d and send an external ray $R(\theta)$ to a horizontal line $R(\theta)$ of imaginary part $2\pi\theta i$, and send an η-equipotential to a vertical line of real part η.
Fix $\eta, s > 0$,

- In $\mathbb{H}_r$, the *log-Böttcher sector* of slope s is and centered at $R(\theta)$ is the sector

$$S(\theta) = \{\rho + i2\pi t \in \mathbb{H}_r \mid |t - \theta| \leq s\rho;\ \rho \leq \eta\}$$

- In $\mathbb{C} \setminus K_P$, the same *log-Böttcher sector* is the sector $S_P(\theta) = \phi_P \circ exp(S(\theta))$.

Note that these in fact are sectors according to the first definition

Remark. We will omit to mention "log-Böttcher" when talking about sectors of polynomial maps.

Proposition 4.2.1. *With the same notations, the opening modulus of a log-Böttcher sector of slope s is proportionate to Arctan$(2\pi s)$ with a proportion constant depending only on d and q.*

Now let S^s_β be a sector with vertex the β-point β_c of Q_c, that is the point of external angle 0 with a slope s to be chosen later. And consider S^s_b to be the preimage sector of S^s_x with vertex $b_c \neq \beta_c$.

Remark. By taking η small enough, we can suppose that S^s_b and S^s_β are disjoint.

Now, we turn back to our construction.

For $i = 0,..,k-1$, Denote by $D'_i = \{G_c < \eta\}$ and $D'_i = \{G_c < 2\eta\}$. Let $U_{i,\beta} = S^s_\beta \cap D_i$, $U_{i,b} = S^s_b \cap D_i$ and $U_i = D \setminus (U_{i,\beta} \cup U_{i,b})$. We define $U'_{i,\beta}, U'_{i,b}$ and U'_i by intersection with D'_i. And let $\tilde{U}_1 := \overline{\overset{\circ}{U_{1,b} \cup U_1}}$

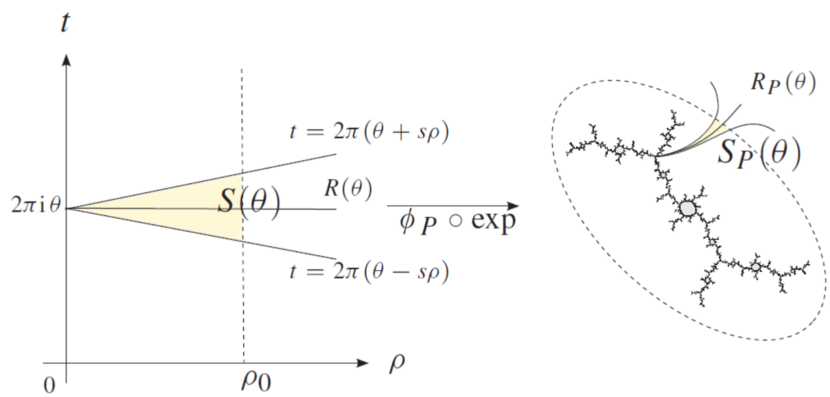


Figure 8: Log-Böttcher sectors, from [BF14]

4.3 Construction

Two different methods are now plausible to do the construction of the quasiconformal map giving the wanted dynamics:

4.3.1 Intertwining surgery

This method was used by A.Epstein and M.Yampolsky in [EY99] in order to fuse the dynamics of two quadratic polynomials in order to create the dynamics of a cubic polynomial and hence embed the product of two limbs of the Mandelbrot set into the real cubic connectedness locus.

The method uses Riemann mappings in order to identify domains of the phase plane of c with others of the pase plane of c_0.

For $i = 0, .., k-1$, for $z = x_i, y_i$, let $S^s(z)$ and $S'^s(z)$ be sectors of slope s around $R(z)$ and $R'(z)$ respectively such that $S'^s(z)$ are send to some $S^s(z)$. And by taking η small enough, we can suppose that these sectors are pairwise disjoint.

Replace now all the sets W_i and V_i by their intersection with the complementary of the sectors. And let $\tilde{V}_1 = Q_{c_0}(V_0)$ and let $\tilde{V}_2 = Q_{c_0}(V_1)$.

The following Riemann mappings do exist:

- $\mathscr{R}_0 : V_0 \to U_0$ such that $\mathscr{R}_0(x_0) = \beta_{c0}$ and $\mathscr{R}_0(y_0) = b_{c0}$.
- $\tilde{\mathscr{R}}_1 : \tilde{V}_1 \to \tilde{U}_1 := \overset{\circ}{\overline{U_{1,b} \cup U_1}}$ such that $\tilde{\mathscr{R}}_1(x_1) = (\beta_{c1})$ and it sends the corners of the equipotential boundary of $\tilde{V}_1$ to those of $\tilde{U}_1$.
- $\tilde{\mathscr{R}}_{1,2} : \tilde{V}_1' \to \tilde{V}_2$ such that $\mathscr{R}_{1,2}(x_1) = x_2$ and it sends the corners of the equipotential boundary of $\tilde{V}_1$ to those of $\tilde{V}_2$.
- for $i = 2, .., k-1$, taking indices modulo k, $\mathscr{R}_{i,i+1} : V_i' \to V_{i+1}$ such that $\mathscr{R}_{i,i+1}(x_i) = x_{i+1}$ and $\mathscr{R}_{i,i+1}(y_i) = y_{i+1}$.

Start by defining

$$f(z) = \begin{cases} \tilde{\mathscr{R}}_1^{-1} \circ Q_c \circ \mathscr{R}_0(z) & \text{if } z \in V_0' \\ \mathscr{R}_{i,i+1}(z) & \text{if } z \in V_i' \text{ for } i = 2, .., k-1 \\ \tilde{\mathscr{R}}_{1,2}(z) & \text{if } z \in \tilde{V}_1' \\ Q_{c_0}(z) & \text{if } z \in \bigcup_i W_i' \setminus \tilde{V}_1' \end{cases}$$

One now only needs to extend f to the different sectors into a quasiconformal way. This should be done if one can properly choose the Riemann mappings and the slopes of the different sectors. And if this is not possible, there exists a notion of trans-quasiconformal surgery introduced in Chapter nine of [BF14] which may relax the condition.

If it was possible to extend f to all of D', it would be a K-quasiregular mapping for some $K > 0$, and one can define an almost complex structure σ by

$$\sigma = \begin{cases} (f^n)^*\sigma_0 & \text{on } f^{-n}(D) \\ \sigma_0 & \text{elsewhere} \end{cases}$$

Well-defined almost complex structure since the only non holomorphic part of f is defined on the sectors which are forward invariant by construction. Moreover, σ is of finite dilatation equal to K^2. Hence, we can apply the intergarbility theorem 1.3.1 to get a quasiconformal map $\phi : \mathbb{D} \to V$ such that $\phi^*\sigma_0 = \sigma$. Take $P = \phi^{-1} \circ f \circ \phi$. $P * \sigma_0 = \sigma_0$ and by Weyl's lemma (theorem 1.2.3), P is holomorphic and hence polynomial-like. Finally, by the straightening theorem 2.4.2, we obtain a complex number $\tilde{c} \in \mathscr{M}$ such that $P \sim_{hyb} Q_{\tilde{c}}$.

Unfortunately, I wasn't able to find the good conditions to extend f quasiconformally!!!

Furthermore, in order to prove that we get $\tilde{c} \in \mathscr{M}_0$, we need to prove the continuity of the map $c \to \tilde{c}$. However, even if f can be constructed continuously on c, the inverse of the integral map ϕ^{-1} may not depend continuously on c, and moreover, the straightening process may not be continuous (see III-2 [DH85]).

4.3.2 Gluing

This method was used by B.Branner and J.H.Hubbard in [BH88] in order to create a new branch of the Julia set of a quadratic polynomial, and hence embed a limb of the Mandelbrot set of periodicity q into another limb of periodicity $q + 1$.

The method starts by constructing a new abstract Riemann surface by gluing domains of c_0's phase plane with domains of the k copies of c's phase plane.

Gluing. The gluing functions will be the Green functions G_{c_0} and G_c. Since the sectors in the c phase plane have transverse sectors with respect to the equipotentials, the identification is possible. For W an element of the decomposition of the c_0 plane and U and element of the decomposition of the c plane, define the equivalence relation $\sim_x$ by

$$x \sim_x \beta_c$$
$$z_1 \in R(x) \sim_x z_2 \in \partial_L U_\beta \Leftrightarrow G_{c_0}(z_1) = G_c(z_2)$$
$$z_1 \in R'(x) \sim_x z_2 \in \partial_R U_\beta \Leftrightarrow G_{c_0}(z_1) = G_c(z_2)$$

And define the equivalence relation $\sim_y$ by the same relation when x is replaced by y and β_c by b_c.

Let X be the surface created by:

- Deleting $\tilde{V}_1$ which has only one neighbourhood W, and replacing it by Gluing $\tilde{U}$ according to the equipotential curves. We obtain $\tilde{U}_1 \cup W/\sim_x$.
- for $i = 2, .., k-1$, deleting V_i which is neighbouring W_x at x_i, and W_y at y_i, and replacing it by U_i by gluing it. We obtain $(W_y \cup V_i/\sim_y) \cup W_x/\sim_x$.

The final result X is endowed with the quotient Riemann structure, and is conformally isomorphic to $\mathbb{D}$. Define X' in the same way by gluing U' and W'.

Construction of a discontinuous f. We may now define the function $f : X \to X'$. Let

$$f(z) = \begin{cases} Q_c(z) & : U'_0 \to \tilde{U}_1 \\ Id(z) & : U'_i \to U'_{i+1} \\ Q_{c_0}(z) & \text{for } z \in Q_{c_0}^{-1}(D) \cap (\bigcup_i W_i \setminus \tilde{V}_1) \\ ??? & \text{elsewhere} \end{cases}$$

The set where f still needs to be defined is the set of preimages of the V_i located in W_i. An idea is to create a sequence of quasiconformal mappings (f_n) where we define at every step to a level of preimages $Q_{c_0}^{-n}(V_i)$ by interpolating, of by defining new Riemann mappings inside these sectors and interpolating them with W_i where f is already defined. This must be done while controlling the increasing rate of the constant of quasiconformality of f_n. Prove that this sequence converges to a limit function. An almost complex structure defined by such a function won't have a finite dilatation, but if the area of the divergence of this dilatation is controlled, the integrability theorem will be still relevant (see Chapter.9 of [BF14]).
But I had no more time to test this idea and to do the calculations.

If we suppose having constructed the wanted f, we can continue in the following way.

Quasiregularization of f. In order to get a quasiconformal mapping, consider sectors around every boundary where f is non continuous. It is possible to set these sectors as in the preceding construction such that they become forward invariant under f.

For the sector S around the boundary between $\tilde{U}_1$ and its neighbouring W. f is continuous on equipotential curves, and is hence quasiconformal.

For sectors around the boundary between some U' and W', f sends the η-equipotential to the η-equipotential in $f(U')$ and to the 2η-equipotential in $f(W')$. Hence, we must glue the 2η-equipotential in $U' \cap S$ to the η-equipotential in $W' \cap S$ in order to define a new X'. X must be extended also by gluing the 4η-equip in U to the 2η-equip in W inside the sector S. The extension can only be in the exterior part of S since a quasiconformal interpolation would automatically do the gluing in S'. Do so in a $\mathscr{C}^2$ way for all sectors.
Now, let $g : \partial_{out} S \to \partial_{out} f(S)$ be a $\mathscr{C}^2$-diffeomorphism. Note that on the other boundaries, f is a near translation equal to Q_{c_0} or Q_c. Hence, by theorem 1.4.4, there exist a quasiregular extension of f inside the sector.

Almost complex structure. Now, we can define an almost complex structure using f by

$$\sigma = \begin{cases} (f^n)^* \sigma_0 & \text{on } f^{-n}(X) \\ \sigma_0 & \text{elsewhere} \end{cases}$$

Well-defined, but its dilatation is controlled, and as assumed in the discontinuous construction step, the Transquasiconformal version of the integrability theorem can be applied to give a quasiconformal map $\phi : X \to \mathbb{D}$. Consider $P = \phi{-}1 \circ f \circ \phi$, and prove that it is conformal. The straightening map gives the final constant $\tilde{c}$.

Continuity of the mapping. We have defined a mapping $\Psi : c \to \tilde{c}$. But the construction process is discontinuous. However, if one proves that the assumed construction of f, it is possible to prove the continuity of Ψ by doing two separate studies on Hyperbolic components and on $\partial\mathscr{M}$ by adapting several proofs present in Chapter.II of [DH85].

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