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# Thèse de doctorat

Discipline : Mathématiques

présentée par

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# Récurrences des sous-variétés lagrangiennes et des solutions de viscosité sous des actions symplectiques qui dévient la verticale

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dirigée par Marie-Claude Arnaud et Jacques Féjoz

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حِبَاؤُكَ والعِيسُ العِتاقُ كأَنَّها　　هِجَانُ المها تَحْدي عليها الرَّحائلُ
فإنْ تَكُ قد ودَّعتَ غيرَ مذّمَّمٍ　　أواهيَ مُلْكٌ ثَبَّتَتْها الأوائلُ
فلا تَبْعَدَنْ إنَّ المنيةَ موعِدٌ　　وكُلُّ امرئٍ يومًا به الحالُ زائلُ
فَما كانَ بَينَ الخَيرِ لَو جاءَ سالِماً　　أَبو حُجُرٍ إِلّا لَيالٍ قَلائِلُ

النابغة الذبياني

# Remerciements

Je tiens à exprimer ma reconnaissance la plus sincère à mes deux directeurs de thèse pour m'avoir introduit au monde de la recherche en mathématiques. À Marie-Claude Arnaud, pour les multiples discussions et explications qui ont nourri ma réflexion, ainsi que pour la liberté dont j'ai pu bénéficier dans l'exploration des idées de cette thèse. Sa patience et son soutien ont été précieux tout au long de ces années, et la rigueur qu'elle m'a enseignée a profondément influencé ma manière d'écrire mes raisonnements et d'exprimer mes idées. À Jacques Féjoz, dont l'enthousiasme et les conseils avisés ont été une source d'inspiration constante. Nos discussions m'ont permis de canaliser et de restructurer mes idées, tout en prenant du recul sur les différents travaux que j'ai menés. Je les remercie tous les deux pour leurs nombreuses relectures ainsi que leur exigence en matière de rédaction et d'exposés. Pour leur gentillesse, leur encouragement et leur soutien dans les moments difficiles.

Je tiens également à remercier les rapporteurs de cette thèse, Ludovic Rifford et Ke Zhang, qui m'ont fait l'honneur de rapporter cette thèse et qui ont pris le temps d'examiner mon travail avec attention. Je suis tout aussi honoré d'avoir un jury exceptionnel. À chacun de ses membres, Nalini Anantharaman, Albert Fathi, Anna Florio et Claude Viterbo, ainsi qu'à Patrick Bernard, invité du jury, je souhaite exprimer ma profonde gratitude pour leur présence lors de la soutenance.

Ma première rencontre avec Nalini Anantharaman remonte à un week-end mathématique au château de Goutelas, un moment décisif qui a orienté mes intérêts vers l'interaction entre les systèmes dynamiques, la géométrie et les équations différentielles. Cet événement fut une première étape essentielle qui, des années plus tard, m'a conduit à découvrir la théorie KAM faible.

Je remercie Albert Fathi, dont les écrits sur cette même théorie ont été une véritable révélation et ont suscité en moi un profond intérêt pour ce domaine. Son travail a joué un rôle central dans mon parcours et dans les choix scientifiques qui ont guidé cette thèse.

Ma reconnaissance va également à Anna Florio, dont la bienveillance et l'ouverture ont grandement contribué à me mettre à l'aise dans le milieu académique. Je la remercie tout particulièrement de m'avoir invité à rejoindre son groupe de travail, où j'ai pu faire de belles rencontres tout en approfondissant mes connaissances.

Un immense merci à Claude Viterbo, qui m'a permis de terminer cette thèse dans de bonnes conditions et m'a accordé sa confiance pour un premier postdoc. Assister à son cours d'introduction à la topologie symplectique pendant l'année de pré-thèse a ravivé mon intérêt pour cette discipline et m'a donné l'envie d'en explorer davantage les richesses.

Enfin, je suis reconnaissant envers Patrick Bernard, dont les travaux ont été une source d'inspiration essentielle pour l'ensemble des recherches menées dans cette thèse. Son influence a été déterminante et a nourri chacun des axes que j'ai explorés.

Cette thèse a également été marquée par des rencontres humaines et professionnelles particulièrement enrichissantes. Je tiens à remercier tout particulièrement Simon Allais, qui a joué le rôle d'un grand frère tout au long de ces années. Sans lui, les deux premières années de cette thèse auraient eu un goût très différent. Je lui suis profondément reconnaissant pour tous les échanges précieux que nous avons partagés et pour tout ce qu'il m'a appris, tant sur le plan humain que mathématique.

Je tiens également à remercier Maxime Zavidovique et Jean-Michel Roquejoffre, avec qui j'ai eu la chance de discuter de mes travaux, confronter mes idées et enrichir ma vision de la discipline. Vos conseils et vos discussions ont été d'une grande aide et ont nourri ma réflexion en élargissant mon champ de vision.

Plusieurs mathématiciens et mathématiciennes, rencontrés au fil des conférences, ont grandement contribué à ma motivation. Ma reconnaissance va tout particulièrement aux membres de l'ANR Cosydy, que j'ai eu le plaisir de rencontrer lors des différentes réunions. Ces rencontres ont constitué mes premiers contacts avec la communauté scientifique et ont été l'occasion de présenter mes premiers exposés sur les résultats de ma thèse. Je remercie les organisateurs et organisatrices de ces événements, ainsi que Adrien, Valentine, Andréa, Francesco et Vincent pour leur soutien, leurs invitations, ainsi que pour nos discussions scientifiques et plus informelles, qui ont rendu ces moments encore plus agréables.

Je souhaite maintenant adresser mes remerciements à mes collègues doctorants ou post-doctorants que j'ai eu le plaisir de rencontrer au fil de ces années. Je remercie César Romero Mora, avec qui je partage des sujets adjacents et avec qui j'ai eu des échanges intéressants autour de ma thèse et de la sienne. Je tiens également à exprimer ma gratitude envers les différents doctorants rencontrés dans les groupes de travail, notamment Nelson, Odylo, Baptiste, Salambo et Dustin, croisés à Jussieu, Dauphine et au Symplectix. Un mot spécial va à Ibrahim Trifa, avec qui je partage la quasi-totalité de mes passions en dehors des mathématiques.

Je n'oublie pas d'adresser un mot d'encouragement à Viktor Maeght, mon petit frère de thèse, avec qui j'ai eu la chance d'échanger sur divers sujets, et à qui je souhaite une excellente continuation pour la suite de son aventure.

Je ne peux passer sous silence les compagnons de bureau qui ont partagé mon quotidien durant ces années. À mes co-bureaux, Camille, Dorian, Pierre, Francesca, Qiu, et Estelle, un immense merci pour ces moments de travail, de rire et de partage. Vos idées, votre énergie et, parfois, simplement votre présence ont été des piliers essentiels dans ce marathon, me chargeant de motivation lorsque j'en avais le plus besoin.

En dehors des quatre murs de ce bureau, il y avait toujours un monde chaleureux pour décompresser, partager des moments de joie ou de détresse autour d'un instant gourmand et convivial. Je pense à Razvan, Romain, Maxime, Juan, Corentin, Elie, et à tous les autres doctorants assidus qui, chaque fin d'après-midi, apportaient leur énergie et leur bonne humeur à la salle commune, rendant ces moments de partage toujours agréables et ressourçants.

Et comme la vie n'est pas que mathématique, mes pensées se tournent vers mes cercles en dehors de l'académique. À mes amis, que j'ai rencontrés pendant ces années à Paris. Aux poireaux chantants, qui m'en aurait posé des défis, et chez qui résonne, à certaines périodes de l'année, la formule « à demain ». À mon groupe du jeu de rôle écrit, avec qui nous avons créé un monde dans un monde, par le récit et la rédaction, et avec qui nous avons noué un lien fort malgré nos localisations géographiques éloignées. Merci à vous tous d'avoir contribué, par votre bienveillance et vos folies, à la stabilité de mes deux équilibres.

Et ma famille à qui je dois tout. Cette thèse s'est inscrite dans un contexte difficile, marqué par une épreuve que la vie décida. À mon père, sans qui je ne serais là aujourd'hui, et dont l'héritage intellectuel et humain continue de me guider.

Enfin, un remerciement tout particulier à ma nouvelle épaule. À toi, Nour, dame à l'épithète féline dont l'irruption me dodeline. Merci à toi d'être et d'avoir été. Et à ta famille, pour son accueil chaleureux et bienveillant.

# Résumé


L'équation de Hamilton-Jacobi joue un rôle fondamental en systèmes dynamiques, en contrôle optimal et en géométrie symplectique. Dans les années 1990, Albert Fathi a introduit la théorie KAM faible, établissant un lien entre la théorie des viscosités de M.G. Crandall et P.L. Lions et la théorie d'Aubry-Mather. Il démontre par cette approche que dans le cadre des Hamiltoniens convexes sur les fibres, dits de Tonelli, les solutions de viscosité sont engendrées par l'opérateur de Lax-Oleinik.

Dans le cas autonome, Fathi montre la convergence du semi-groupe de Lax-Oleinik $\mathcal{T}^t$, entraînant ainsi la convergence de toute solution de viscosité vers une solution stationnaire, appelée solution KAM faible. Cependant, cette convergence ne se généralise pas au cas non-autonome. Nous nous intéressons alors aux solutions de viscosité limites, qui forment l'ensemble non-errant $\Omega(\mathcal{T}^1)$ de l'opérateur de Lax-Oleinik. Nous observons que, du fait que cet opérateur est 1-lipschitz, ces solutions non-errantes sont en réalité récurrentes. Cette thèse est consacrée à l'étude des solutions de viscosité récurrentes.

Nous étudions d'abord l'action de la restriction de $\mathcal{T} = \mathcal{T}$ à cet ensemble $\Omega(\mathcal{T})$ et constatons que les principales propriétés des solutions KAM faibles se généralisent à ces solutons récurrentes, ce qui en fait un ensemble naturel à considérer dans le cas non-autonome. Nous caractérisons également ces éléments de $\Omega(\mathcal{T})$ comme étant les solutions de viscosité globales et bornées. De plus, nous établissons une formule de représentation qui décrit $\Omega(\mathcal{T})$ , analogue à celle connue pour l'ensemble $\mathrm{Fix}(\mathcal{T})$ des solutions KAM faibles.

Par la suite, et afin de justifier l'intérêt de considérer l'ensemble non-errant, nous construisons un hamiltonien de Tonelli pour lequel $\Omega(\mathcal{T})$ contient une solution récurrente mais non périodique. Nous arrivons même à faire en sorte que le Hamiltonien construit permette l'existence d'une telle solution de régularité $C^\infty$.

Enfin, dans un dernier chapitre, nous généralisons un résultat théorème de Birkhoff multidimensionnel sur les sous-variétés Lagrangiennes invariantes, dû à M.-C. Arnaud et A. Venturelli, aux sous-variétés Lagrangiennes récurrentes. Plus précisément, nous considérons une sous-variété lagrangienne $\mathcal{L}$, Hamiltoniennement isotope à la section nulle, dont les images $\phi_H^n(\mathcal{L})$ par un flot Hamiltonien de Tonelli admettent deux sous-suites de convergentes en temps positifs et négatifs, avec un contrôle sur leurs longueurs. Nous montrons alors que cette sous-variété Lagrangienne et toutes ses images sont des graphes $C^1$ des différentielles spatiales d'une solution de viscosité récurrente de $\Omega(\mathcal{T})$.

## Mots-clés

Équation de Hamilton-Jacobi, Solutions de viscosité, Théorie KAM-faible, Théorie d'Aubry-Mather, Semi-groupe de Lax-Oleinik, Fonctions génératrices, Sous-variétés lagrangiennes.

# Recurrence of Lagrangian submanifolds and viscosity solutions under symplectic actions that deviate the vertical

## Abstract

The Hamilton-Jacobi equation plays a fundamental role in dynamical systems, optimal control, and symplectic geometry. In the 1990s, Albert Fathi introduced weak KAM theory, establishing a connection between the viscosity theory of M.G. Crandall and P.L. Lions and Aubry-Mather theory. Through this approach, he demonstrated that for fiberwise convex Hamiltonians, known as Tonelli Hamiltonians, viscosity solutions are generated by the Lax-Oleinik operator.

In the autonomous case, Fathi proved the convergence of the Lax-Oleinik semigroup $\mathcal{T}^t$, which consequently implies the convergence of every viscosity solution to a stationary solution, called a weak KAM solution. However, this convergence does not extend to the non-autonomous case. We therefore focus on limit viscosity solutions, which form the non-wandering set $\Omega(\mathcal{T}^1)$ of the Lax-Oleinik operator. We observe that, by non-expansiveness of this operator, these non-wandering solutions are actually recurrent solutions. This thesis is dedicated to the study of recurrent viscosity solutions.

We first analyze the action of the restriction of $\mathcal{T}$ to its non-wandering set $\Omega(\mathcal{T})$ and find that the main properties of weak KAM solutions extend to these recurrent solutions, making it a natural set to consider in the non-autonomous case. We also characterize the elements of $\Omega(\mathcal{T})$ as the global and bounded viscosity solutions. Furthermore, we establish a representation formula describing $\Omega(\mathcal{T})$, analogous to the one known for the set $\mathrm{Fix}(\mathcal{T})$ of weak KAM solutions.

Next, to justify the relevance of considering the non-wandering set, we construct a Tonelli Hamiltonian for which $\Omega(\mathcal{T})$ contains a recurrent but non-periodic solution. Moreover, we ensure that the constructed Hamiltonian admits such a solution with $C^\infty$ regularity.

Finally, in the last chapter, we generalize a multidimensional Birkhoff theorem on invariant Lagrangian submanifolds, due to M.-C. Arnaud and A. Venturelli, to recurrent Lagrangian submanifolds. More precisely, we consider a Lagrangian submanifold $\mathcal{L}$, Hamiltonianly isotopic to the zero section, whose images $\phi_H^n(\mathcal{L})$ under a Tonelli Hamiltonian flow admit two convergent subsequences in both positive and negative times, with controlled lengths. We then show that this Lagrangian submanifold and all its images are $C^1$ graphs of the spatial differentials of a recurrent viscosity solution in $\Omega(\mathcal{T})$.

## Keywords

Hamilton-Jacobi Equaton, Viscosity Solutions, Weak-KAM Theory, Aubry-Mather Theory, Lax-Oleinik Semigroup, Generating Functions, Lagrangian Submanifolds.

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# Première partie

# Introduction

# Chapitre 1

# Un point historique

La dynamique hamiltonienne est une branche de la mécanique classique qui décrit l'évolution temporelle des systèmes physiques en termes de positions et de moments conjugués, plutôt qu'en termes de forces comme dans la mécanique newtonienne. Elle est fondée sur la notion de fonction Hamiltonienne $H$, une fonction qui encapsule toute l'énergie du système, à savoir l'énergie cinétique et potentielle, et qui détermine les équations de mouvement du système.

Dans notre cadre, nous considérons une variété $M$ compacte, connexe, sans bords, de fibré cotangent $T^*M$. Ce fibré cotangent est naturellement muni d'une *forme de Liouville* $\lambda$ définie par

$$\lambda(x,p) = p \circ d\pi(x) = p.dx$$

où $\pi : T^*M \to M$ est la projection. Le fibré cotangent $T^*M$ est alors muni d'une forme symplectique standard $\omega = -d\lambda = dx \wedge dp$ en coordonnées locales.

Un *Hamiltonien* $H$ est une application $H : T^*M \to \mathbb{R}$ auquel nous associons le champs de vecteurs $X_H$ définit par l'équation

$$\iota_{X_H}\omega := \omega(X_H, \cdot) = dH$$

Le flot Hamiltonien $\phi_H^t$ associé à ce champ de vecteurs donne alors les équations du mouvement du système, qui s'écrivent en coordonnées locales comme les équations hamiltoniennes suivantes : si l'on note pour tout $(x,p) \in T^*M$, $\phi_H^t(x,p) = (x(t), p(t))$, nous avons le système

$$\begin{cases} \dot{x}(t) = \partial_p H(x(t), p(t)) \\ \dot{p}(t) = -\partial_x H(x(t), p(t)) \end{cases} \qquad \text{avec} \begin{cases} x(0) = x \\ p(0) = p \end{cases} \tag{1.0.1}$$

L'étude de ces systèmes dynamiques conservatifs, en mécanique classique et dans le cadre de la dynamique hamiltonienne, a occupé une place centrale dans la compréhension des structures et comportements à long terme des systèmes complexes comme les systèmes planétaires, les particules en interaction, ou les mouvements des fluides incompressibles. Les flots et les difféomorphismes hamiltoniens, qui régissent ces systèmes, offrent un cadre riche pour explorer des phénomènes allant de la régularité au chaos. Dès les premières investigations, les mathématiciens ont cherché à décrire comment ces systèmes évoluent au fil du temps, notamment en termes de stabilité et d'instabilité des trajectoires.

Un objectif fondamental dans l'étude de l'évolution à long terme de ces systèmes est l'identification des sous-ensembles invariants, tels que les tores invariants, les orbites périodiques ou les ensembles chaotiques. Ceux-ci offrent une perspective sur la géométrie globale du système dynamique. Par exemple, dans les systèmes hamiltoniens, les tores invariants peuvent structurer l'espace des phases $T^*M$ en couches régulières, organisant ainsi les trajectoires. De plus, l'identification de ces ensembles permet de distinguer les zones où le comportement du système devient imprévisible, contribuant à une représentation géométrique complète ou partielle du système.

## 1.1 Le cas intégrable et la théorie KAM

Dans l'étude de ces systèmes dynamiques, une première approche serait d'étudier des systèmes dynamiques proches de l'intégrable. Les systèmes intégrables sont des modèles idéaux qui possèdent suffisamment d'intégrales du mouvement (ou quantités conservées) pour rendre leurs évolutions prévisibles. Plusieurs définitions non-équivalentes existent pour la notion d'intégrabilité. Une définition formelle de la plus forte d'entre-elles serait la suivante.

**Définition 1.1.1.** Sur une variété $M$ de dimension $d$, on dit qu'un Hamiltonien $H_0 : T^*M \to \mathbb{R}$ est *intégrable* s'il existe $d-1$ Hamiltoniens $H_1, .., H_{d-1} : T^*M \to \mathbb{R}$ tels que

1. les Hamiltoniens $H_i$ sont en involution, c'est-à-dire pour tous entiers $0 \leq i, j \leq d-1$, le crochet de Poisson de $H_i$ et $H_j$ s'annule

$$\{H_i, H_j\} := \omega(X_{H_i}, X_{X_{H_j}}) = 0$$

2. Les différentielles $d_x H_i$ sont linéairement indépendantes en tout point $x$ de $M$.

Une avancée majeure dans cette direction est le théorème d'Anorl'd-Liouvillle [Lio55, Arn63c, Arn89] qui montre que la configuration intégrable n'est possible que dans le cas du tore $d$-dimensionnel $M = \mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$, et que dans ce cas, l'espace des phases $T^*\mathbb{T}^d = \mathbb{T}^d \times \mathbb{R}^d$ est feuilleté par des tores invariants où les trajectoires sont quasi-périodiques, décrites en termes de variables d'angle-action.

Pour aller plus loin, on peut s'intéresser aux systèmes qui sont proches du cas intégrable en adoptant une approche perturbative. L'étude de l'impact des petites perturbations sur les trajectoires et la stabilité des systèmes intégrables s'est avérée particulièrement difficile et a donné naissance à la théorie KAM, une des avancées les plus marquantes du 20ème siècle en systèmes dynamiques hamiltoniens.

La théorie KAM, développée par Kolmogorov [Kol54], puis perfectionnée par Arnold [Arn63a, Arn63b] et Moser [Mos62] traite de la persistance des tores invariants dans les systèmes hamiltoniens légèrement perturbés. Elle démontre que pour certaines valeurs des fréquences de rotation mal approximées par les rationnels (appelées fréquences diophantiennes), les tores quasi-périodiques invariants sont robustes face aux perturbations, tandis que d'autres valeurs peuvent conduire à des comportements plus complexes, voire chaotiques. La transition vers le chaos survient dans des zones du système où les tores invariants disparaissent, laissant place à des trajectoires erratiques et imprévisibles.

Néanmoins, John N. Mather [Mat84] démontre que ces tores ne survivent pas aux grandes perturbations, ce qui nécessite de rechercher d'autres approches pour identifier des ensembles invariants pour des systèmes non-intégrables plus généraux.

## 1.2 La théorie de Birkhoff sur les courbes invariantes

Un premier pas vers une étude globale (non-perturbative) des tores invariants fut la théorie de Birkhoff. Dans le cas de dimension 1, G.D. Birkhoff s'est intéressé à des difféomorphismes du cylindre qui préservent les aires et dévient les verticales. De telles applications sont appelées twists du cylindre et sont définies comme suit

**Définition 1.2.1.** Un difféomorphisme $f = (Q, P) : T^*\mathbb{T}^d \to T^*\mathbb{T}^d$ est un *twist symplectique* si

1. (Symplectique) il est symplectique, i.e. $f^*\omega = \omega$.
2. (Twist) Le relevé de l'application $(q, p) \mapsto (q, Q(q, p))$ est un difféomorphisme du revêtement $\mathbb{R}^{2d}$ de $T^*\mathbb{T}^d$.

Dans le cas du cylindre $d = 1$, cette condition est équivalente à dire que l'image de la verticale $\{x\} \times \mathbb{R}$ est un graphe (Lipschitz) au-dessus du revêtement $\mathbb{R}$ du cercle $\mathbb{T}^1$. Le twist est positif ou négatif selon le sens de cette déviation à droite ou à gauche.

Dans le cas du cylindre $T^*\mathbb{T}^1$, Birkhoff [Bir22] démontre qu'une courbe $\gamma$ invariante par le twist $f$ du cylindre, et essentielle, c'est-à-dire non-homotope à un point, est un graphe Lipschitz au-dessus du cercle $\mathbb{T}^1$.

Ce théorème a servi de fondement à de nombreux travaux dans les années 1980, visant soit à le généraliser aux twists en dimensions supérieures, soit à l'étendre à des ensembles invariants discontinus qui généralisent les tores KAM.

Les bons ensembles invariants à considérer en dimensions supérieures sont les sous-variétés Lagrangiennes $\mathscr{L}$, c'est-à-dire les sous-variétés $\mathscr{L}$ de dimension $d$ et sur lesquelles la forme symplectique standard $\omega$ s'annule. En effet, les courbes en dimension $d = 1$ et les tores KAM invariants de dimension maximale trouvés par la théorie KAM sont des sous-variétés Lagrangiennes.

Ainsi, la question à se poser est de savoir quand une sous-variété Lagrangienne $\mathscr{L}$ invariante sous l'action d'un twist symplectique est un graphe au-dessus de la base $\mathbb{T}^d$. Plusieurs conditions suffisantes ont été établies dans [Her89, BP92, DCR23], impliquant une hypothèse topologique, selon laquelle $\mathscr{L}$ doit être homotope à la base $\mathbb{T}^d$, ainsi qu'une hypothèse dynamique requérant souvent que la dynamique du twist restreinte à $\mathscr{L}$ soit récurrente par chaînes.

Ces résultats restent aussi valables pour les flots Hamiltoniens générés par des Hamiltoniens convexes sur les fibres, dits de Tonelli (Définition 2.0.1). Ces flots $\phi_H^t$ sont des twists symplectiques (exacts) pour des temps très petits, mais cessent de l'être pour des temps grands. Ils sont aussi définis sur des variétés générales compactes connexes $M$ qui permettent de généraliser ces résultats hors du cadre des tores $\mathbb{T}^d$. Une avancée importante dans la question fut le théorème de Marie-Claude Arnaud [Arn10] qui a montré qu'une sous-variété Lagrangienne $\mathscr{L}$ hamiltoniennement isotope à la section nulle de $T^*M$, (c'est-à-dire $\mathscr{L} = \varphi(0_{T^*M})$ pour le temps 1 $\varphi = \varphi^1$ d'un flot Hamiltonien quelconque), et invariante par un flot Hamiltonien de Tonelli $\phi_H$ est alors un graphe au-dessus de la section nulle $0_{T^*M}$ de $T^*M$. Ce résultat a été ensuite généralisé pour des sous-variétés moins régulières [BdS12, AOdS18] ou au cas de Hamiltoniens non-autonomes $H(t, x, p)$ dépendant du temps [AV17]. Nous allons détailler les idées de la preuve de base dans la section 2.3, pour ensuite l'étendre dans le chapitre 9 à des sous-variétés récurrentes pour un type de convergence que l'on introduit.

D'autres généralisations, inspirées par la théorie d'Aubry-Mather et la théorie KAM faible, ont donné naissance à des ensembles dynamiques plus larges qui étendent les tores KAM. Ces ensembles, en accord avec les théorèmes de Birkhoff, sont des graphes Lipschitz (partiels ou complets) au-dessus de la base $M$. Nous allons les explorer dans les prochaines sections.

## 1.3 La théorie d'Aubry-Mather

Une avancée majeure vers l'étude des systèmes dynamiques non-intégrables est la théorie d'Aubry-Mather, développée dans les années 1980 par Serge Aubry [ALD83] et John N. Mather [Mat82] qui traite par une approche variationnelle de l'existence d'orbites quasi-périodiques et organisées dans des configurations globalement non-intégrables, souvent de faible dimension comme le modèle du pendule double, et même en dehors du cadre de régularité requis par KAM.

Cette théorie fut d'abord appliquée à des modèles discrets, comme le modèle de Frenkel-Kontorova, qui étudie un réseau de particules (ou de spins) soumis à des interactions non linéaires. Elle s'étend également aux twists du cylindre étudiés par Birkhoff (définition 1.2.1).

Dans notre cadre Hamiltonien, un principe variationnel existe et consiste à minimiser une action Hamiltonienne. Cette action s'exprime de manière particulièrement adaptée à la théorie d'Aubry-Mather si l'on suppose que le Hamiltonien est convexe et surlinéaire sur les fibres, dits de Tonelli (Définition 2.0.1). Un tel Hamiltonien $H$ peut être associé à un Lagrangien $L(x,v) : TM \to \mathbb{R}$ défini cette fois sur le fibré tangent de $M$ par

$$L(x,v) = \max_{p \in T_x^* M} \{p(v) - H(x,p)\} \tag{1.3.1}$$

Cette quantité est le conjugué convexe du Hamiltonien, et elle est souvent interprétée comme la différence entre l'énergie cinétique et l'énergie potentielle. L'action à minimiser est donnée par l'action Lagrangienne

$$A(\gamma) = \int_0^t L(\gamma(\tau), \dot{\gamma}(\tau))\, d\tau$$

où $\gamma : [0,t] \to M$ est une courbe absolument continue liant deux extrémités préalablement choisies $\gamma(0) = x$ et $\gamma(t) = y$.

Dans les deux exemples qui précèdent, Aubry et Mather ont démontré que même si le système n'est pas intégrable, il existe des ensembles invariants formés d'orbites minimisantes. Ces orbites ne sont pas nécessairement périodiques, mais elles restent confinées dans une région délimitée de l'espace des phases. Elles minimisent une action variationnelle et constituent ce que l'on appelle des *ensembles d'Aubry-Mather*. Ils sont souvent totalement discontinus et ne sont ni totalement chaotiques ni entièrement réguliers, mais exhibent un comportement intermédiaire.

Par exemple, dans le cadre du twist du cylindre, pour chaque nombre de rotation réel

$\rho$, il existe un ensemble d'Aubry-Mather $\mathcal{M}_\rho$ qui présente les caractéristiques suivantes (voir [Ban88, Gol01])

- Si $\rho = p/q$ est rationnel, l'ensemble $\mathcal{M}_{p/q}$ est une union d'orbites périodiques de nombre de rotation $p/q$, reliées par d'autres orbites hétéroclines.
- Si $\rho$ est irrationnel, l'ensemble $\mathcal{M}_\rho$ est soit un ensemble de Cantor, soit une courbe invariante

De plus, ces ensembles vérifient un théorème de Birkhoff (voir la section 1.2 qui précède) et sont des graphes Lipschitz au-dessus de leur projection sur le cercle. Ils sont ordonnés verticalement, selon un ordre croissant ou décroissant, en fonction de leur nombre de rotation.

Ce type de configuration se généralise aux dimensions supérieures, où l'on observe des ensembles dits *cantoriques*. Ces ensembles ne forment pas des tores réguliers, mais sont des ensembles de Cantor dispersés dans l'espace des phases tout en étant organisés selon des structures fines, et contenus dans des tores non-invariants.

Ainsi, en introduisant les concepts d'orbites minimisantes et d'ensembles cantoriques, la théorie d'Aubry-Mather a permis de comprendre comment des structures invariantes subsistent dans des systèmes perturbés au-delà du cadre intégrable. Cela a ouvert de nouvelles perspectives sur la dynamique des systèmes fortement non linéaires et chaotiques.

## 1.4 Les mesures minimisantes de Mather

Dans les années 1990, John N. Mather [Mat91, MF94] a généralisé les travaux antérieurs en introduisant la théorie des mesures minimisantes. Cette approche, également basée sur un principe variationnel, est une extension de la théorie d'Aubry-Mather sur les trajectoires minimisantes, mais elle est formulée dans un cadre plus général et statistique. Mather a proposé qu'au lieu de se concentrer uniquement sur des trajectoires individuelles, on pouvait introduire des mesures de probabilité sur l'espace des phases qui minimisent l'action. Ces mesures minimisantes, appelées aujourd'hui *mesures de Mather*, offrent une nouvelle perspective sur le comportement dynamique, en tenant compte de la répartition des trajectoires minimisantes dans l'ensemble du système.

Dans ses travaux, Mather considère un Hamiltonien de Tonelli (Définition 2.0.1) $H(x,p): T^*M \to \mathbb{R}$, associé un Lagrangien $L(x,v): TM \to \mathbb{R}$ également de Tonelli. Et il s'intéresse aux mesures de probabilité invariantes par le flot Lagrangien $\phi_L$ (qui génère les courbes minimisantes), et qui minimisent l'action parmi cette famille de mesures

$$A_L(\mu) = \int_{TM} L(x,v)\, d\mu(x,v)$$

Il prouve que de telles mesures existent et que grâce à la surlinéarité du Lagrangien $L$ demandée dans les hypothèses Tonelli, ces mesures sont toutes confinées dans un compact du fibré tangent $TM$. Ainsi, il obtient que la réunion de ces supports noté $\mathcal{M}$ est un ensemble compact non-vide de $TM$. Cet ensemble est appelé *ensemble de Mather*.

Le théorème de Carneiro [Car95] affirme que cet ensemble est inclus dans un niveau d'énergie $\{E = \alpha_0\}$ du Lagrangien où la fonction énergie $E : TM \to \mathbb{R}$ est définie par

$$E(x,v) = H \circ \mathcal{L}(x,v) = H(x, \partial_v L(x,v))$$

avec $\mathcal{L} : T^*M \to TM$ est la transformée de Legendre qui conjugue le flot Hamiltonien $\phi_H$ au flot Lagrangien $\phi_L$ (voir le chapitre 2). Cette constante d'énergie $\alpha_0$ s'appelle la *valeur critique de Mañé* . Et pour toute mesure minimisante $\mu$, Mather obtient la relation

$$-\alpha_0 = \int_{TM} L \, d\mu = \inf \left\{ \int_{TM} L \, d\nu \ ; \ \nu \text{ invariant} \right\}$$

Enfin, Mather démontre un théorème de type Birkhoff pour l'ensemble $\tilde{\mathcal{M}}$ (voir la section 1.2). Comme pour les ensembles d'Aubry-Mather, il établit que $\tilde{\mathcal{M}}$ est un graphe Lipschitz au-dessus de sa projection sur la variété $M$.

En résumé, cet ensemble $\tilde{\mathcal{M}}$ capture les informations données par les mesures de probabilités invariantes par le flot Lagrangien. Ces dernières généralisent les orbites minimales périodiques ou quasi-périodiques des systèmes intégrables à des systèmes plus complexes et permettent de capturer les propriétés statistiques de trajectoires minimisantes.

Cependant, ces informations ne concernent que le niveau d'énergie $\alpha_0$. Ainsi, pour généraliser cet ensemble, Mather considère pour toute 1-forme fermée $\eta$ le Lagrangien modifié $L_\eta : TM \to \mathbb{R}$ défini par

$$L_\eta(x,v) = L(x,v) - \eta_x(v) \tag{1.4.1}$$

possède le même flot Lagrangien que $L$. Notons que le Hamiltonien $H_\eta : T^*M \to \mathbb{R}$ associé est

$$\begin{aligned} H_\eta(x,p) &= \max_{v \in T_x M} \{p(v) - L(x,v) + \eta_x(v)\} \\ &= \max_{v \in T_x M} \{(p + \eta_x)(v) - L(x,v)\} \\ &= H(x, p + \eta_x) \end{aligned}$$

En remarquant que l'action $A_{L_\eta}$ ne dépend que de la classe de cohomologie $[\eta] \in H^1(M,\mathbb{R})$ de $\eta$, il est possible de considérer l'ensemble de Mather $\tilde{\mathcal{M}}_{[\eta]}$ associé à $L_\eta$ et contenu dans le niveau d'énergie $\alpha_{[\eta]}$, qui ne dépend que de cette classe de cohomologie $[\eta]$. De cette manière, il définit la *fonction alpha de Mather* $\alpha : H^1(M,\mathbb{R}) \to \mathbb{R}$ donnée par

$$\alpha([\eta]) = \alpha_{[\eta]} = -\inf_{\mu}\left\{\int_{TM} L_\eta \, d\mu\right\} \tag{1.4.2}$$

où l'infimum est pris sur les mesures minimisantes de $L_\eta$.

Cette application $\alpha$ se révèle convexe et surlinéaire, et on peut donc lui associer un conjugué convexe $\beta : H_1(M,\mathbb{R}) \to \mathbb{R}$ appelé *fonction bêta de Mather* et défini par

$$\beta(h) = \max_{[\eta]\in H^1(M,\mathbb{R})}\left\{\langle[\eta],h\rangle - \alpha_{[\eta]}\right\} \tag{1.4.3}$$

où $\langle\cdot,\cdot\rangle$ désigne le crochet de dualité. Pour mieux comprendre cette fonction, Mather l'interprète en termes de vecteurs de rotation qui quantifient la moyenne asymptotique du déplacement dans l'espace des phases d'un système dynamique. Plus précisément, il est possible d'associer à toute mesure invariante $\mu$, le vecteur $\rho(\mu) \in H_1(M,\mathbb{R})$ tel que pour toute classe de cohomologie $[\eta] \in H^1(M,\mathbb{R})$, le crochet de dualité se lit comme suit

$$\langle[\eta],\rho(\mu)\rangle = \int_{TM} \eta \, d\mu$$

Et inversement, il prouve que pour toute classe d'homologie $h \in H_1(M,\mathbb{R})$, il existe une mesure invariante $\mu$ de vecteur de rotation $\rho(\mu) = h$. Ceci lui permet de réécrire la formule (1.4.3) sous la forme

$$\beta(h) = \inf\left\{\int_{TM} L \, d\mu \; ; \; \rho(\mu) = h\right\} \tag{1.4.4}$$

qui donne donc le minimum de l'action à vecteur de rotation fixé. Notons par $\tilde{\mathcal{M}}^h \subset TM$ la réunion des supports des mesures réalisant l'infimum dans (1.4.4).

Cette interprétation permet de faire le lien entre ces ensembles minimisants à vecteurs de rotation fixés et les différents objets apparus dans les travaux antérieurs. Dans le cas proche de l'intégrable, Mather établit une correspondance avec les tores quasi-périodiques de la théorie KAM, qui possèdent des fréquences $h$. Quant aux twists du cylindre, il identifie ces ensembles aux ensembles d'Aubry-Mather associés au nombre de rotation $h$. De plus, l'étude de la régularité de ces fonctions alpha et bêta révèle des informations supplémentaires sur les formes et les intersections possibles entre ces différents ensembles $\tilde{\mathcal{M}}_c$ et $\tilde{\mathcal{M}}^h$ lorsque $c$ et $h$ varient [Mot22].

Ces idées furent les fondements de la théorie KAM-faible d'Albert Fathi qui révèlera par la suite le lien profond entre la théorie des mesures minimisantes et la théorie des solutions de viscosité de Michael G. Crandall et Pierre-Louis Lions pour l'équation de Hamilton-Jacobi.

# Chapitre 2

# Théorie KAM-faible

On se place dans le cadre des Hamiltoniens de Tonelli $H : T^*M \to \mathbb{R}$ sur le fibré cotangent d'une variété compacte connexe $M$. Nous définissons proprement cette notion initialement introduite dans les travaux de Mather [Mat91].

**Définition 2.0.1.** Un Hamiltonien $H(x,p) : T^*M \to \mathbb{R}$ est dit de *Tonelli* s'il satisfait les hypothèses suivantes :

— Régularité : $H$ est de classe $\mathcal{C}^2$.

— Convexité stricte : $\partial_{pp}H(x,p) > 0$ pour tout $(x,p) \in T^*M$.

— Surlinéarité : $H(x,p)/|p| \to \infty$ quand $|p| \to \infty$, uniformément en $x \in M$.

Sous ces hypothèses, on peut associer à $H(x,p)$ un *Lagrangien* de Tonelli $L(x,v)$ : $TM \to \mathbb{R}$ défini par la conjugaison convexe

$$L(x,v) = \max_{p \in T_x^*M} p(v) - H(x,p) \tag{2.0.1}$$

ce qui donne symétriquement

$$H(x,p) = \max_{v \in T_xM} p(v) - L(x,v) \tag{2.0.2}$$

Ces identités fournissent l'inégalité de Fenchel

$$p(v) \leq H(x,p) + L(x,v) \quad \text{avec égalité si et seulement si } p = \partial_v L(x,v) \tag{2.0.3}$$

Le Lagrangien fournit un cadre variationnel où l'on s'intéresse aux courbes $\gamma : [0,t] \to M$ qui minimisent l'action

$$A_L(\gamma) = \int_0^t L(\gamma(\tau), \dot{\gamma}(\tau)) \, d\tau \tag{2.0.4}$$

et qui permettent de mesurer le *potentiel* $h_0^t : M \times M \to \mathbb{R}$ défini par

$$h_0^t(x,y) = \inf \left\{ A_L(\gamma) \;\middle|\; \begin{array}{rl} \gamma : & [0,t] \to M \\ & 0 \mapsto x \\ & t \mapsto y \end{array} \right\} \tag{2.0.5}$$

où l'infimum est pris sur les courbes absolument continues $\gamma$.

Cet infimum est en fait un minimum réalisé par des courbes minimisantes qui vérifient l'équation d'Euler-Lagrange donnée par

$$\frac{d}{dt}\big(\partial_v L(\gamma(t),\dot\gamma(t))\big) = \partial_x L(\gamma(t),\dot\gamma(t))$$

Cette équation génère le *flot Lagrangien*, ou *flot d'Euler-Lagrange* $\phi_L$ associé à $L$.

La dynamique Hamiltonienne générée par le flot Hamiltonien $\phi_H$ et la dynamique Lagrangienne générée par le flot Lagrangien $\phi_L$ sont conjugués par la transformation de Legendre $\mathcal{L} : TM \to T^*M$ qui est un difféomorphisme défini par

$$\mathcal{L}(x,v) = (x, \partial_v L(x,v)) \quad \text{et} \quad \mathcal{L}^{-1}(x,p) = (x, \partial_p H(x,p))$$

Noter que cette transformation est celle qui donne l'égalité dans l'inégalité de Fenchel (2.0.3).

La théorie KAM faible, développée par Albert Fathi [Fat97b, Fat08] dans les années 1990, est une extension naturelle de la théorie des mesures minimisantes de Mather qui établit un lien avec la théorie des solutions de viscosité [Lio82, CL83]. Fathi revient ici à l'étude plus classique des courbes minimisantes et utilise celles-ci pour identifier des sous-ensembles intéressants du fibré tangent $TM$, invariants par le flot, et qui généralisent dans des cas régulier les ensembles d'Aubry-Mather et les tores KAM. Ces ensembles sont des graphes de différentielles de solutions de viscosité (Lipschitziennes) $u : M \to \mathbb{R}$ de l'équation de Hamilton-Jacobi

$$H(x, d_x u) = \alpha \tag{2.0.6}$$

pour une constante $\alpha$ qu'il identifiera.

Dans un cadre Tonelli proche de l'intégrable, si $\mathscr{L}$ est un tore KAM invariant, un théorème de Birkhoff multidimensionnel dû à Arnaud [Arn10] (voir sections 1.2 et 2.3.2) affirme que $\mathscr{L}$ est le graphe au-dessus de $M$ d'une forme différentielle $\eta = c + du$ de classe cohomologique $c = [\eta] \in H^1(M,\mathbb{R})$. De plus, il est connu par Hermann [Her89] que ces tores KAM (de dimensions maximales) sont des sous-variétés Lagrangiennes dont l'invariance par le flot implique qu'ils sont inclus dans un niveau d'énergie $H = \alpha$. Il en résulte que

$u : M \to \mathbb{R}$ est une solution régulière de l'équation de Hamilton-Jacobi

$$H_c(x, d_x u) = H(x, c + d_x u) = \alpha$$

Ceci explique le lien entre l'équation de Hamilton-Jacobi et la théorie KAM. Cette dernière fournit des solutions au sens classique et régulier à cette équation. La théorie KAM faible cherche à généraliser ce fait en proposant des solutions faibles, donc moins régulières, ce qui justifie le nom de la théorie. Il apparaît a posteriori que les solutions définies par Fathi coïncident, dans ce cadre convexe où elles sont définies, avec la notion de solutions de viscosité déjà bien connue en théorie des EDPs.

Par ailleurs, Fathi démontre aussi que la constante $\alpha$ est en réalité égale à $\alpha(c) = \alpha_c$, donnée par la fonction alpha de Mather. La fréquence associée à la dynamique quasi-périodique du tore KAM $\mathscr{L}$ est alors déterminée par l'unique vecteur de rotation $h \in H_1(M, \mathbb{R})$ vérifiant $\alpha(c) + \beta(h) = \langle c, h \rangle$. Fathi tisse ainsi un lien étroit avec la théorie des mesures invariantes de Mather, qu'il approfondira en soulignant l'importance de l'ensemble de Mather dans le comportement de ses solutions KAM-faibles.

Dans cette section, nous présentons une description plus détaillée de la théorie KAM faible, de ses concepts fondamentaux, ainsi que de ses principaux résultats. Nous nous concentrerons ensuite sur la description de ces solutions KAM-faibles à travers une formule de représentation établie par G. Contreras [Con01]. Enfin, nous conclurons par l'une des applications de cette théorie, qui a permis de démontrer un théorème multidimensionnel de Birkhoff sur les sous-variétés Lagrangiennes exactes invariantes.

## 2.1 Théorie KAM-faible de Fathi

Afin de définir les solutions faibles de l'équation (2.0.6) pour $\alpha = 0$, Fathi réintroduit le semi-groupe de Lax-Oleinik $(\mathcal{T}_0^t)_{t>0}$ défini sur l'ensemble $\mathcal{C}(M, \mathbb{R})$ des fonctions scalaires continues sur $M$ par

$$\begin{aligned} \mathcal{T}_0^t u(x) &= \inf_{\substack{\gamma : [0,t] \to M \\ t \mapsto x}} \left\{ u(\gamma(0)) + \int_0^t L(\gamma(\tau), \dot{\gamma}(\tau))\, d\tau \right\} \\ &= \inf_{y \in M} \left\{ u(y) + h_0^{s,t}(y, x) \right\} \end{aligned} \tag{2.1.1}$$

et pour obtenir un opérateur pour toute constante $\alpha$, il le translate comme suit

$$\mathcal{T}_\alpha^t u = \mathcal{T}_0^t u + \alpha.t \tag{2.1.2}$$

Pour toute constante réelle $\alpha \in \mathbb{R}$, on dit qu'une fonction $u : M \to \mathbb{R}$ est une $\alpha$-sous-solution de l'équation (2.0.6), et on note $u \prec L + \alpha$, si pour toute courbe $\gamma : [0, t] \to M$, on a

$$u(\gamma(t)) - u(\gamma(0)) \leq \int_0^t L(\gamma(\tau), \dot{\gamma}(\tau))\, d\tau + \alpha.t \tag{2.1.3}$$

On vérifie que l'assertion $u \prec L + \alpha$ est équivalente à

$$u(x) \leq \mathcal{T}_\alpha^t u(x) \tag{2.1.4}$$

Si on considère l'ensemble $\mathcal{H}(\alpha)$ des $\alpha$-sous-solutions, il découle du fait que $L$ est minoré, que pour $\alpha \ll -1$ suffisamment petit, on a $\mathcal{H}(\alpha) = \varnothing$. Fathi démontre à posteriori que la constante critique de Mañé , $\alpha_0$ vérifie

$$\alpha_0 = \inf\{\alpha \; ; \; \mathcal{H}(\alpha) \neq \varnothing\} \tag{2.1.5}$$

Enfin, il déduit des propriétés du semi-groupe de Lax-Oleinik $\mathcal{T}_{\alpha_0}^t$ que les fonctions de $\mathcal{H}(\alpha_0)$ sont équi-lipschitziennes, que $\mathcal{H}(\alpha_0)$ est fermé dans $\mathcal{C}(M, \mathbb{R})$ et que pour tout $t > 0$, l'opérateur $\mathcal{T}_{\alpha_0}^t$ est faiblement contractant (ou 1-Lipschitz) pour la norme $\| \cdot \|_\infty$ et laisse $\mathcal{H}(\alpha_0)$ invariant. Une application judicieuse du théorème du point fixe de Banach donne alors

**Théorème 2.1.1** (KAM faible)**.** *Il existe une fonction $u_- : M \to \mathbb{R}$ telle que pour tout $t > 0$,*

$$\mathcal{T}_{\alpha_0}^t u_- = u_- \tag{2.1.6}$$

Une telle fonction $u_-$ est appelée solution KAM faible de l'équation de Hamilton-Jacobi critique

$$H(x, d_x u) = \alpha_0 \tag{2.1.7}$$

Nous notons $\mathcal{S}^-$ l'ensemble des solutions KAM-faibles de cette équation.

Ce théorème d'existence montre que l'infimum définissant $\alpha_0$ dans (2.1.5) est en fait un minimum atteint par une ou des $\alpha_0$-sous-solutions que l'on appelle sous-solutions critiques.

Les propriétés régularisantes du semi-groupe de Lax-Oleinik $\mathcal{T}_\alpha^t$ montrent que les éléments de $\mathcal{S}^-$ sont équi-Lipschitz. Fathi montre même que pour toute condition initiale $u_0$ et tous temps $t > 0$, l'application $\mathcal{T}^t u_0$ est semi-concave, c'est-à-dire somme d'une application $C^\infty$ régulière et d'une application concave. En particulier, les solutions KAM-faibles $u_- \in \mathcal{S}^-$ sont semi-concaves.

Mieux que ça, Fathi s'intéresse aux propriétés asymptotiques du semi-groupe de Lax-

Oleinik $\mathcal{T}_\alpha^t$ et remarque qu'une égalité de la forme

$$\mathcal{T}_\alpha^t u = u$$

pour un temps strictement positif $t > 0$, ne se réalise que pour la valeur critique de Mañé $\alpha = \alpha_0$. Plus largement, les applications $u_\alpha(t,x) = \mathcal{T}_\alpha^t u(x)$ ont une croissance asymptotique linéaire de la forme $(\alpha - \alpha_0).t$. Ceci donne une nouvelle caractérisation de $\alpha_0$.

**Proposition 2.1.1.** *Les assertions suivantes sont équivalentes*

1. $\alpha = \alpha_0$.
2. *Il existe une application scalaire* $u \in \mathcal{C}(M,\mathbb{R})$ *telle que la famille* $(\mathcal{T}_\alpha^t u)_{t>0}$ *est bornée dans* $\mathcal{C}(M,\mathbb{R})$.
3. *Pour toute application scalaire* $u \in \mathcal{C}(M,\mathbb{R})$, *la famille* $(\mathcal{T}_\alpha^t u)_{t>0}$ *est bornée dans* $\mathcal{C}(M,\mathbb{R})$.

Dans cette étude, il montre que les applications $u_\alpha(t,x) = \mathcal{T}_\alpha^t u(x)$ sont en fait des solutions de viscosité de l'équation de Hamilton-Jacobi non-stationnaire

$$\partial_t u + H(x, d_x u) = \alpha \tag{2.1.8}$$

Et que dans le cas critique $\alpha = \alpha_0$, elles convergent toutes vers des solutions KAM faibles de l'équation stationnaire associée

**Théorème 2.1.2** (Convergence vers KAM faible)**.** *Pour toute application scalare* $u \in \mathcal{C}(M,\mathbb{R})$, *la famille* $(\mathcal{T}_{\alpha_0}^t u)_{t>0}$ *converge vers une solution KAM faible* $u_\infty$ *de l'équation de Hamilton-Jacobi stationnaire* (2.1.7).

Par conséquent, si on note $\mathrm{Fix}(\mathcal{T}_{\alpha_0}^t)$ l'ensemble des applications scalaires fixées par l'opérateur $\mathcal{T}_{\alpha_0}^t$, alors le théorème montre que pour tout temps $t > 0$

$$\mathcal{S}^- = \mathrm{Fix}(\mathcal{T}_{\alpha_0}^t) = \bigcap_{s>0} \mathrm{Fix}(\mathcal{T}_{\alpha_0}^s)$$

Nous nous sommes focalisés sur les sous-solutions critiques de $\mathcal{H}(\alpha_0)$ et réalisant les inégalités sur l'opérateur de Lax-Oleinik (2.1.4) et sur l'action des courbes (2.1.3). Et nous avons défini les solutions KAM faibles comme étant les solutions réalisant l'égalité l'opérateur de Lax-Oleinik (2.1.6). Fathi donne une interprétation des solutions KAM-faibles en terme d'égalités sur l'action des courbes. Pour ceci, il introduit la notion de courbes calibrées.

Soit $u \prec L + \alpha_0$. On dit qu'une courbe $\gamma : I \subset \mathbb{R} \to M$ est *u-calibrée* ou *calibrée par u* si elle vérifie pour tous $s < t$ dans $I$

$$u(\gamma(t)) - u(\gamma(s)) = \int_s^t L(\gamma(\tau)\dot{\gamma}(\tau))\, d\tau + \alpha_0.(t-s)$$

De telles courbes calibrées sont minimisantes pour l'action Lagrangienne. Fathi montre qu'une solution KAM faible $u_-$ est exactement une fonction $u_-$ telle que pour tout point $x$, il existe une courbe $u_-$-calibrée $\gamma_x : (-\infty, 0] \to M$ telle que $\gamma_x(0) = x$.

De plus, en analysant l'inégalité (2.1.4), il prouve que ces courbes réalisent le cas d'égalité dans l'inégalité de Fenchel (2.0.3). Ceci donne la...

**Proposition 2.1.2.** *Soit $u_- : M \to \mathbb{R}$ une solution KAM-faible et $\gamma : I \to \mathbb{R}$ une courbe $u$-calibrée définie sur un intervalle ouvert $I$. Alors pour tout $t \in I$, $u_-$ est différentiable en $x = \gamma(t)$ et*

$$(x, d_x u_-(x)) = (x, \partial_v L(\gamma(t), \dot\gamma(t))) = \mathcal{L}(\gamma(t), \dot\gamma(t)) \tag{2.1.9}$$

Observons l'implication d'une telle proposition sur la différentielle $du_-$ d'une solution KAM-faible $u_-$. Les courbes $u_-$-calibrées $\gamma_x$ sont minimisantes, et suivent donc la trajectoire du flot Lagrangien $\phi_L$. On obtient pour tout $t < 0$

$$(\gamma_x(t), d_{\gamma_x(t)} u_-) = \mathcal{L}(\gamma_x(t), \dot\gamma_x(t)) = \mathcal{L} \circ \phi_L^t(x, d_x u_-) = \phi_H^t(x, d_x u_-)$$

Ceci implique que le graphe $\mathcal{G}(d_x u_-) = \{(x; d_x u_-) \ ; \ d_x u_- \text{ existe}\}$ de $du_-$ dans $T^*M$ est invariant en temps négatifs par le flot Hamiltonien $\phi_H$, i.e. pour tout $t < 0$,

$$\phi_H^t\big(\mathcal{G}(d_x u_-)\big) \subset \mathcal{G}(d_x u_-)$$

et par continuité du flot on a aussi pour tout $t < 0$,

$$\phi_H^t\left(\overline{\mathcal{G}(d_x u_-)}\right) \subset \overline{\mathcal{G}(d_x u_-)}$$

Considérons les ensembles

$$\tilde{\mathcal{A}}^*_{u_-} = \bigcap_{t<0} \phi_H^t\left(\overline{\mathcal{G}(d_x u_-)}\right) \subset T^*M \quad \text{et} \quad \tilde{\mathcal{A}}_{u_-} = \mathcal{L}^{-1}(\tilde{\mathcal{A}}^*_{u_-}) \subset TM \tag{2.1.10}$$

$\tilde{\mathcal{A}}_{u_-}$ est en fait l'union de $(\gamma, \dot\gamma)$ sur les courbes globales $\gamma : \mathbb{R} \to M$ calibrées par $u_-$. Cet ensemble est appelé *ensemble d'Aubry relatif à $u_-$* et peut être interprété comme le lieu de différentiabilité de la solution KAM-faible $u_-$. Il est compact et invariant par le flot Lagrangien. De plus, il vérifie un théorème de Birkhoff (section 1.2) et est est un graphe partiel bi-lipschitzien au-dessus de sa projection $\mathcal{A}_{u_-}$ sur $M$. En effet, le seul point au-dessus d'un point $x \in \mathcal{A}_{u_-}$ est, comme donné par la proposition 2.1.2, le point $(x, v) = \mathcal{L}^{-1}(x, d_x u_-)$.

Plus généralement, Fathi définit les ensembles $\tilde{\mathcal{A}}$ et $\tilde{\mathcal{N}} \subset TM$ par

$$\tilde{\mathcal{A}} = \bigcap_{u_- \in \mathcal{S}^-} \tilde{\mathcal{A}}_{u_-} \quad \text{et} \quad \tilde{\mathcal{N}} = \bigcup_{u_- \in \mathcal{S}^-} \tilde{\mathcal{A}}_{u_-} \tag{2.1.11}$$

Le premier ensemble $\tilde{\mathcal{A}}$ s'appelle l'ensemble d'Aubry, et le second $\tilde{\mathcal{N}}$ s'appelle l'ensemble de Mañé . L'un est formé par les courbes globales calibrées par toutes les solutions KAM-faibles. Et l'autre contient toute courbe globale calibrée par au moins une de ces solution KAM-faible.

Il découle ainsi des propriétés des $\tilde{\mathcal{A}}_{u_-}$ que $\tilde{\mathcal{A}}$ et $\tilde{\mathcal{N}}$ sont des ensembles compacts invariants par le flot Lagrangien $\phi_L$. De plus, $\tilde{\mathcal{A}}$ est un graphe partiel bi-lipschitzien au dessus de sa projection $\mathcal{A}$ sur $M$ formé par les points $\mathcal{L}^{-1}(x, d_x u_-)$ pour n'importe quelle solution $u_-$. Ce dernier théorème de Birkhoff n'est cependant pas vérifié par l'ensemble de Mañé $\tilde{\mathcal{N}}$.

Finalement, Fathi lie les ensembles dynamiques importants de sa théorie par les inclusions successives suivantes

$$\tilde{\mathcal{M}} \subset \tilde{\mathcal{A}} \subset \tilde{\mathcal{N}} \subset \mathcal{E}$$

où $\mathcal{E} = \mathcal{L}^{-1}\big(\{H = \alpha_0\}\big)$ est le niveau d'énergie critique, vu dans le fibré tangent $TM$.

L'ensemble d'Aubry, introduit initialement par Mather, est en fait une meilleure généralisation des tores KAM que l'ensemble de Mather $\tilde{\mathcal{M}}$ [Mat91, FGS09]. En effet, quand ces deux ensembles ne coïncident pas, $\tilde{\mathcal{A}}$ est plus étendu et approxime toujours les propriétés des tores KAM.

## 2.2 La vision de Mañé et Formule de Représentation

### 2.2.1 Vision de Mañé

Le lien entre les différents ensembles dynamiques introduits dans la théorie KAM-faible a été exploré par Ricardo Mañé [Mn97, CDI97] qui donne une nouvelle interprétation des courbes globales calibrées en tant que courbes statiques ou semi-statiques, c'est-à-dire en tant que courbes qui permettent un retour à leur point initial en dépensant un minimum d'effort.

Dans ce but, on se place dans le cas critique $\alpha = \alpha_0$ dans l'équation de Hamilton-Jacobi (2.1.7), et on simplifie la notation du semi-groupe de Lax-Oleinik par $\mathcal{T} = \mathcal{T}_{\alpha_0}$. On considère le potentiel modifié $h^t : M \times M \to \mathbb{R}$ défini par

$$\begin{aligned} h^t &= h_0^t + \alpha_0 t \\ &= \inf \left\{ A_{L+\alpha_0}(\gamma) \;\middle|\; \begin{array}{rl} \gamma : & [0,t] \to M \\ & 0 \mapsto x \\ & t \mapsto y \end{array} \right\} \end{aligned} \tag{2.2.1}$$

Et on définit les deux potentiels qui minimisent l'action indépendamment du temps.

Le premier est le *potentiel de Mañé* $m : M \times M \to \mathbb{R}$ donné par

$$m = \inf_{t>0} h^t$$

Et le second est une barrière initialement introduite par Mather [Mat93], nommée la *barrière de Peierls* $h^\infty : M \times M \to \mathbb{R}$ donnée par

$$h^\infty = \liminf_n h^n$$

Par la définition (2.1.2) du semi-groupe de Lax-Oleinik $\mathcal{T}$ qui s'écrit

$$\mathcal{T}^t u(x) = \inf_{y \in M} \{u(y) + h^t(y, x)\}$$

et par la proposition 2.1.1 qui énonce qu'il est borné pour les temps positifs, on déduit que ces barrières sont finies, et donc bien définies. De plus, le théorème de convergence 2.1.2 de Fathi montre que la barrière de Peierls $h^\infty$ est en fait une limite

$$h^\infty = \liminf_n h^n = \lim_{t \to +\infty} h^t \tag{2.2.2}$$

et un calcul donne que pour tout point $x \in M$, l'application $h^\infty(x, \cdot)$ est une solution KAM-faible de l'équation de Hamilton-Jacobi. Cette barrière de Peierls donne alors une famille explicite de solutions KAM-faibles qui s'avèrera très importante pour les décrire toutes.

Mañé s'intéresse aux courbes absolument minimisantes pour l'action de $L+\alpha_0$. Il définit les *courbes semi-statiques* comme étant les courbes $\gamma : I \to \mathbb{R}$ vérifiant pour tous $s < t$ dans $I$

$$A_{L+\alpha_0}(\gamma_{|[s,t]}) = \int_s^t L(\gamma(\tau), \dot{\gamma}(\tau))\, d\tau + \alpha_0(t-s) = m(\gamma(s), \gamma(t))$$

et les *courbes statiques* comme étant les courbes vérifiant

$$A_{L+\alpha_0}(\gamma_{|[s,t]}) = \int_s^t L(\gamma(\tau), \dot{\gamma}(\tau))\, d\tau + \alpha_0.(t-s) = -m(\gamma(t), \gamma(s))$$

Il est à noter que les courbes statiques sont aussi des courbes semi-statiques. Plus précisément, une courbe $\gamma : [s, t] \to M$ est statique si et seulement si elle est semi-statique et vérifie

$$m(\gamma(t), \gamma(s)) + m(\gamma(t), \gamma(s)) = 0$$

En d'autres termes, ce sont des courbes pour lesquelles il est possible de revenir à des points déjà traversés en fournissant très peu d'effort. Ceci justifie leur nomination de courbes statiques.

Lorsque l'on regarde une courbe $\gamma : [s,t] \to M$ calibrée par une solution KAM-faible $u_- \in \mathcal{S}^-$, on peut voir que la domination $u \prec L + \alpha_0$ implique que la courbe $\gamma$ minimise absolument l'action dans le sens où toute courbe $\sigma : [a,b] \to M$ liant $\gamma(s)$ à $\gamma(t)$ vérifie

$$\int_s^t L(\gamma(\tau),\dot\gamma(\tau))\,d\tau + \alpha_0.(t-s) = u(\gamma(t)) - u(\gamma(s)) \leq \int_s^t L(\sigma(\tau),\dot\sigma(\tau))\,d\tau + \alpha_0.(t-s)$$

Il en résulte que $A_{L+\alpha_0}(\gamma_{|[s,t]}) = m(\gamma(s),\gamma(t))$ et que la courbe $\gamma$ est semi-statique. Plus généralement, les ensembles d'Aubry $\tilde{\mathcal{A}}$ et de Mañé $\tilde{\mathcal{N}}$ étudiés dans la théorie KAM-faible peuvent être définis en termes de ces courbes globalement statiques et semi-statiques. En effet,

$$\begin{aligned}\tilde{\mathcal{A}} &= \{(x,v) \in TM \mid \pi \circ \phi_L^t(x,v) : \mathbb{R} \to M \text{ est statique}\}\\ \tilde{\mathcal{N}} &= \{(x,v) \in TM \mid \pi \circ \phi_L^t(x,v) : \mathbb{R} \to M \text{ est semi-statique}\}\end{aligned} \tag{2.2.3}$$

Une exploration du lien entre le potentiel de Mañé $m$ et la barrière de Peierls $h^\infty$ révèle la formule

$$h^\infty(x,y) = \inf_{z \in \mathcal{A}} \{m(x,z) + m(z,y)\}$$

où l'ensemble $\mathcal{A}$ est l'ensemble d'Aubry projeté sur $M$. En particulier, elle montre que si $x$ est un point de $\mathcal{A}$, alors

$$h^\infty(x,x) = 2m(x,x) = 0$$

La réciproque est tout aussi vraie, et donne la caractérisation de l'ensemble d'Aubry projeté $\mathcal{A}$ comme réunion des points statiques

$$\mathcal{A} = \{x \in M \,;\, h^\infty(x,x) = 0\} \tag{2.2.4}$$

Mañé classe ces points statiques en classes d'équivalence d'une relation d'ordre définie comme suit. Il introduit une semi-distance $d : \mathcal{A} \times \mathcal{A} \to \mathbb{R}$ définie par

$$d(x,y) = h^\infty(x,y) + h^\infty(y,x)$$

Et la relation d'équivalence associée

$$x \sim y \quad\Longleftrightarrow\quad d(x,y) = 0$$

Les classes d'équivalence de cette relation s'appellent les *classes statiques*. On note leur ensemble par $\mathbb{A}$, et le relevé associé dans l'ensemble d'Aubry $\tilde{\mathcal{A}}$ par $\tilde{\mathbb{A}}$.

Ces classes statiques sont compactes connexes dans $\mathcal{A}$. Elles contiennent chacune un élément de l'ensemble de Mather $\mathcal{M}$ et donc peuvent toutes être représentées par un sous-ensemble $\mathbb{M}$ de $\mathcal{M}$.

Mañé montre aussi que si une courbe globale $\gamma$ est semi-statique, alors son $\alpha$-limite $\alpha(\gamma)$ et son $\omega$-limite $\omega(\gamma)$ sont incluses dans l'ensemble d'Aubry $\mathcal{A}$ et plus précisément, chacune appartient à une seule classe statique de $\mathbb{A}$. Sur ces idées, Mañé prouve aussi le...

**Théorème 2.2.1.** *1. L'ensemble d'Aubry $\tilde{\mathcal{A}}$ est récurrent par chaînes par le flot Lagrangien $\phi_L$.*

*2. L'ensemble de Mañé $\tilde{\mathcal{N}}$ est transitif par chaînes par le flot Lagrangien $\phi_L$.*

Résultat amélioré par Contreras et Iturriaga [CI99] dans le cas d'un nombre fini de classes statiques, en introduisant une relation d'ordre partiel $\preccurlyeq$ sur $\mathbb{A}$ définie par

1. $\preccurlyeq$ est réflexif et transitif.
2. Pour $\Lambda_1, \Lambda_2 \in \mathbb{A}$, on a $\Lambda_1 \preccurlyeq \Lambda_2$ s'il existe une courbe globale semi-statique $\gamma$ telle que $\alpha(\gamma) \in \Lambda_1$ et $\omega(\gamma) \in \Lambda_2$.

**Théorème 2.2.2.** *Si l'ensemble des classes statiques $\mathbb{A}$ est fini, alors pour tous $\Lambda_1, \Lambda_2 \in \mathbb{A}$, on a $\Lambda_1 \preccurlyeq \Lambda_2$.*

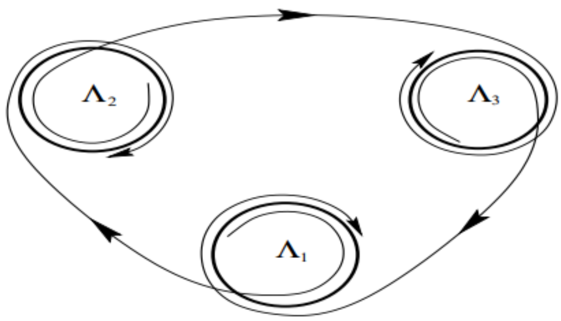



FIGURE 2.1 – Exemple de connexions entre classes statiques.
Figure prise de [CI99].

### 2.2.2 Formule de Représentation

Les interprétations précédentes des ensembles d'Aubry et de Mather permettent une description précise des solutions KAM faibles. La raison en est que ces dernières sont entièrement déterminées par leurs courbes calibrées, qui sont semi-statiques. À partir de cela, Contreras [Con01] a établi une formule de représentation précise des solutions KAM faibles, permettant de les décrire de manière détaillée à partir des classes statiques et des barrières de Peierls $h^\infty$.

Considérons une solution KAM-faible $u_- \in \mathcal{S}^-$ et un point quelconque $x$ de la variété $M$. Il a été établi que cette solution possède une courbe calibrée $\gamma_x : (-\infty, 0] \to M$ arrivant en $\gamma(0) = x$. Comme pour les courbes de l'ensemble de Mañé , celles-ci ont aussi leurs $\alpha$-limites $\alpha(\gamma_x)$ dans une même classe statique de $\mathbb{A}$. Mieux encore, on peut prouver que $\alpha(\gamma_x)$ contient un point de l'ensemble de Mather projeté $\mathcal{M}$. Ce phénomène résulte en un théorème d'unicité des solutions KAM-faibles sur $\mathcal{M}$.

**Théorème 2.2.3** (Unicité sur $\mathcal{M}$)**.** *Si deux solutions KAM-faibles $u$ et $v \in \mathcal{S}^-$ coïncident sur l'ensemble de Mather projeté $\mathcal{M}$, alors elles coïncident partout sur $M$ et $u = v$.*

Par ailleurs, si deux points $x$ et $y$ sont dans une même classe statique $\mathbb{A}$, alors la condition de domination $u \prec L + \alpha_0$ donne

$$u(x) - u(y) \le h^\infty(y,x) \quad \text{et} \quad u(y) - u(x) \le h^\infty(x,y)$$

qui se somment en

$$0 = [u(x) - u(y)] + [u(y) - u(x)] \le d(x,y) = h^\infty(y,x) + h^\infty(x,y) = 0$$

Ainsi, les inégalités sont en fait des égalités et on obtient

$$u(x) = u(y) + h^\infty(y,x)$$

Ainsi, l'image de $x$ est déterminée par l'image de $y$ et on déduit que le théorème d'unicité s'étend sur l'ensemble $\mathbb{M} \subset \mathcal{M}$ des représentants des classes statiques $\mathbb{A}$.

Notons que la barrière de Peierls $h^\infty$ intervient dans cette formule, ce qui laisse présager son importance dans le comportement des solutions KAM-faibles. Jetons un coup d'œil sur ces solutions KAM-faibles particulières de la forme $h^\infty(x,\cdot)$. Un calcul montre qu'une courbe semi-statique $\gamma : (-\infty, 0] \to M$ qui possède $x$ dans son $\alpha$-limite $\alpha(\gamma)$ est calibrée par la solution KAM-faible $h^\infty(x,\cdot)$.

Par conséquent, si $u_-$ est une solution de viscosité quelconque et $x$ un point de $M$ associé à la courbe $u_-$-calibrée $\gamma_x : (-\infty, 0] \to M$, nous avons pour un point $x_\alpha \in \alpha(\gamma_x)$ et pour tous temps $s < t < 0$,

$$u_-(\gamma_x(t)) - u_-(\gamma_x(s)) = A_{L+\alpha_0}(\gamma_{x|[s,t]}) = h^\infty(x_\alpha, \gamma_x(t)) - h^\infty(x_\alpha, \gamma_x(s))$$

et donc que

$$u_-(\gamma_x(t)) - h^\infty(x_\alpha, \gamma_x(t)) = u_-(\gamma_x(s)) - h^\infty(x_\alpha, \gamma_x(s))$$

Il en résulte que $u_-$ et $h^\infty(x_\alpha, \gamma_x(t))$ coïncident à une constante $c_x$ près sur le graphe de la courbe $\gamma_x$, et

$$u_-(\gamma_x(\cdot)) = c_x + h^\infty(x_\alpha, \gamma_x(\cdot))$$

Il s'avère que ces constantes $c_x$ ne dépendent que des classes $y \in \mathbb{M}$ de $x_\alpha$. Et en particulier, pour ces points $y$, on a

$$u_-(y) = c_y + h^\infty(y,y) = c_y$$

De plus, ces constantes sont liées par des inégalités données par la domination de la solution KAM-faibles $u_- \prec L + \alpha_0$. En effet, pour tous $y, z \in \mathbb{M}$,

$$c_y - c_z = u_-(y) - u_-(z) \leq h^\infty(z,y)$$

Ces inégalités définissent la notion de domination pour des application $\psi : \mathbb{M} \to \mathbb{R}$.

**Définition 2.2.4.** Une application $\psi : \mathbb{M} \to \mathbb{R}$ est dite *dominée* si pour tous $y, z \in \mathbb{M}$,

$$\psi(y) - \psi(z) \leq h^\infty(z,y)$$

On note $\mathrm{Dom}(\mathbb{M}, h^\infty)$ l'ensemble de ces applications dominées.

Ainsi, le théorème de représentation est énoncé comme suit

**Théorème 2.2.5** (Formule de Représentation [Con01])**.** *L'application $\Psi$ suivante est une bijection*

$$\begin{array}{rcl} \Psi : \mathrm{Dom}(\mathbb{M}, h^\infty) & \longrightarrow & \mathcal{S}^- \\ \psi & \longmapsto & \inf_{y \in \mathbb{M}} \{\psi(y) + h^\infty(y, \cdot)\} \end{array} \tag{2.2.5}$$

*d'inverse l'application restriction*

$$\begin{array}{rrcl} \Phi : & \mathcal{S}^- & \longrightarrow & \mathrm{Dom}(\mathbb{M}, h^\infty) \\ & u_- & \longmapsto & u_{-|\mathbb{M}} \end{array} \tag{2.2.6}$$

En d'autres termes, les barrières de Peierls sont en quelque sorte le squelette de toutes les solutions KAM-faibles. De plus, le nombre de classes statiques contrôle la dimension de l'ensemble des solutions KAM-faibles $S^-$.

De surcroît, en étudiant l'application $\psi$ associée aux solutions de viscosité $h^\infty(x, \cdot)$ pour $x \in \mathcal{A}$, il est possible de déduire la formule

$$h^\infty(x,y) = \sup_{u_- \in \mathcal{S}^-} \{u_-(y) - u_-(x)\} = \sup_{\substack{u_- \in \mathcal{S}^- \\ u_-(x) = 0}} \{u_-(y)\} \tag{2.2.7}$$

Cette formule était déjà connue, sous une forme plus générale, par Fathi [Fat97a].

### 2.2.3 Application au pendule

**Pendule Simple.** Regardons l'implication de ce résultat dans le cas du pendule simple de Hamiltonien $H : T^*\mathbb{T}^1 \to \mathbb{R}$ de formule

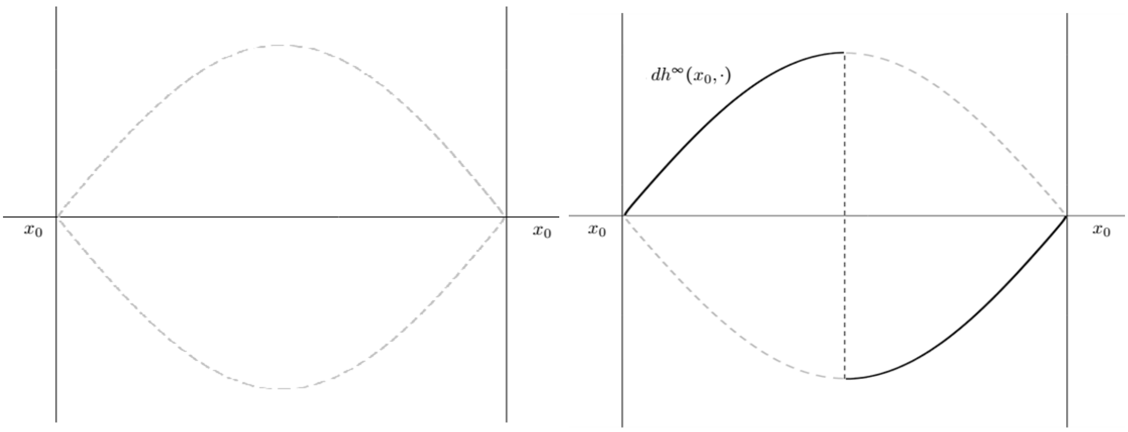
$$H(x,p) = \frac{1}{2}p^2 + V(x) = \frac{1}{2}p^2 + \cos(2\pi x)$$

Il est possible de montrer que la valeur critique de Mañé associée à ce Hamiltonien est donné par

$$\alpha_0 = \max_M V(x) = 1$$

Donc le graphe $du_-$ des solutions KAM-faibles seront dans le niveau d'énergie critique $\{H = 1\}$ du portrait de phase du pendule, représenté dans la figure 2.2a.

La différentielle d'une solution KAM-faible $du_-$ est une forme exacte et semi-concave dans le niveau d'énergie critique $\{H = 1\}$, c'est-à-dire que les points de discontinuité de sa différentielles ne peuvent se déplacer que vers le "bas" dans le portrait de phase. Et en observant le portrait de phase, on déduit qu'il n'existe qu'un seul choix possible pour une telle différentielle, représenté dans 2.2b.




(a) Niveau critique $\{H = 1\}$. (b) Différentielle des solutions KAM-faible.

FIGURE 2.2 – Solutions KAM-faibles du pendule simple.

D'après la formule des représentations, cette unicité est due au fait que l'ensemble de Mather est constitué d'un seul point $\mathcal{M} = \{x_0\}$ avec $x_0 = 0 \in \mathbb{T}^1$. Il en résulte l'existence d'une unique classe statique $\mathbb{M} = \{x_0\}$ et toutes les solutions KAM-faibles sont de la forme

$$u_c(x) = c + h^\infty(x_0, x) \quad , \ c \in \mathbb{R}$$

de différentielle $dh^\infty(x_0,\cdot)$.

**Revêtement double du pendule, ou 2-pendule.** Considérons maintenant le pendule associé au Hamiltonien $H : T^*\mathbb{T}^1 \to \mathbb{R}$ donné par la formule

$$H(x,p) = \frac{1}{2}p^2 + V(2x) = \frac{1}{2}p^2 + \cos(4\pi x)$$

On a toujours $\alpha_0 = 1$, et $\{H = 1\}$ est le revêtement double du niveau critique du pendule simple. La semi-concavité des solutions KAM-faible $u_-$ impose qu'il n'y ait qu'une seule discontinuité possible dans chaque "oeil" du niveau critique. De plus, l'exactitude de la différentielle $du_-$ nécessite que ces points de discontinuités soient symétriques par rapport au point $x_0 = 0$, de sorte que l'aire algébrique délimitée par le graphe de $du_-$ soit nulle. Il en résulte que les solutions KAM-faibles possibles soient celles représentées dans la figure 2.3, et dont le point de discontinuité $x_c$ décrit $[x_0, x_1] = [0, 1/2]$.

Nous voyons ici que l'ensemble des solutions KAM-faibles $\mathcal{S}^-$ est de dimension 2. En effet, il y a une famille à un paramètre de différentielles possibles $du_c$ (représenté dans la figure), et les solutions KAM-faibles sont de la forme

$$u_{c,c'}(x) = c' + \int du_c$$

Ceci est dû au fait que l'ensemble de Mather $\mathcal{M}$ est constitué de deux points $x_0 = 0$ et $x_1 = 1/2$ qui ne sont pas dans la même classe statique. Donc $\mathbb{M} = \{x_0, x_1\}$.

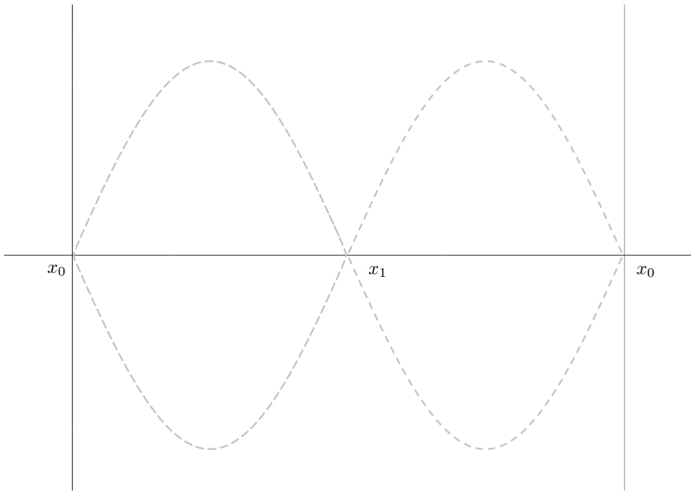
Identifions les différentielles des barrières de Peierls $h^\infty(x_0,\cdot)$ et $h^\infty(x_1,\cdot)$. La formule (2.2.7) dit que ces applications sont les solutions KAM-faibles maximales qui s'annulent respectivement en $x_0$ et $x_1$. Ainsi, afin d'avoir leurs différentielles, il faut donc maximiser leurs différentielles en partant respectivement de $x_0$ et de $x_1$ dans le sens des $x$ croissants. Ceci détermine leurs différentielles représentées dans la figure 2.3c.

Si on regarde la différentielle $du_c$ d'une solution KAM-faible de point de discontinuité $x_c$, on voit bien qu'il y a une région $[-x_c, x_c]$ où $du_c = dh^\infty(x_0,\cdot)$ et une autre région $[x_c, 1 - x_c]$ où $du_c = dh^\infty(x_1,\cdot)$. Ceci explique bien la formule de représentation

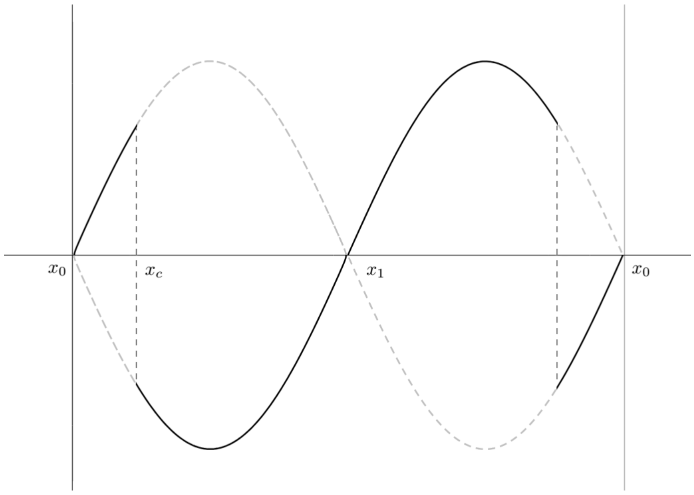
$$u_c(x) = \inf\{u(x_0) + h^\infty(x_0,\cdot), u(x_1) + h^\infty(x_1,\cdot)\}$$

La condition de domination demandée dans la formule assure que chaque élément de l'infimum est réalisé sur une région de la variété $M$. Ces régions peuvent être des points lorsqu'on est dans le bord de $\mathrm{Dom}(\mathbb{M}, h^\infty)$.

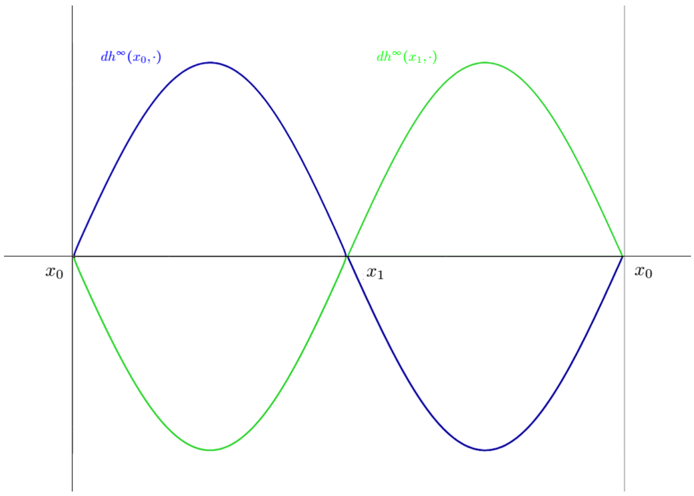
Par ailleurs, nous avons bien $d(x_0, x_1) = h^\infty(x_0, x_1) + h^\infty(x_1, x_0)$ est la somme des aires positives délimitées par l'axe des abscisses et les courbes de $dh^\infty(x_0,\cdot)$ et $dh^\infty(x_1,\cdot)$




(a) Niveau critique $\{H = 1\}$.




(b) Différentielle des solutions KAM-faible.




(c) Différentielle des barrières de Peierls.

FIGURE 2.3 – Solutions KAM-faibles du 2-pendule.

respectivement sur $[x_0, x_1]$ et $[x_1, x_0 + 1]$. Ceci est cohérent avec le fait que $d(x_0, x_1) > 0$ et que les deux points $x_0$ et $x_1$ appartiennent à deux classes statiques différentes.

## 2.3 Solutions variationnelles et théorème de Birkhoff multidimensionnel

Une autre vision géométrique de la théorie KAM-faible prend racine dans l'utilisation d'outils de topologie symplectique. En s'appuyant sur les idées de Chaperon, Sikorav et plus tard, de Viterbo, il est possible de définir des solutions variationnelles de l'équation de Hamilton-Jacobi non-stationnaire (2.1.8) pour des conditions initiales lipschitziennes $u_0$, en utilisant des sélecteurs de graphes de sous-variétés Lagrangiennes. Cette approche offre un cadre pour obtenir des solutions qui, dans le cas où le système est de Tonelli et sous une condition de semi-concavité de la condition initiale, coïncident avec les solutions de viscosité générées par le semi-groupe de Lax-Oleinik $u(t,x) = \mathcal{T}_\alpha^t u_0(x)$ et, par conséquent, convergent vers des solutions KAM-faibles.

Hors du cadre Tonelli, cependant, cette correspondance entre les solutions variationnelles et les solutions de viscosité n'est plus valable, comme en témoignent les travaux de Roos [Roo19] qui mettent en lumière le lien entre les deux théories.

De retour au systèmes Hamiltoniens de Tonelli, cette nouvelle perspective, ajoutée aux idées issues de la théorie KAM-faible, a permis de démontrer certains résultats importants tels que le théorème de Birkhoff multidimensionnel démontré par Marie-Claude Arnaud [Arn10].

Dans cette section, nous exposons d'abord les idées sous-jacentes aux solutions variationnelles de Hamilton-Jacobi, puis nous présentons comment Marie-Claude Arnaud a étendu ces techniques pour établir des liens avec la théorie de Birkhoff.

### 2.3.1 Solutions variationnelles

Afin de coller aux notations habituellement utilisées en topologie symplectique, nous allons utiliser les coordonnées $(q,p)$ au lieu de $(x,p)$ dans le fibré cotangent $T^*M$.

Rappelons qu'une sous-variété Lagrangienne $\mathcal{L}$ est une sous-variété de $T^*M$ de dimension $d = \dim M$ et sur laquelle la forme symplectique $\omega$ s'annule. Une *fonction génératrice* $S : M \times \mathbb{R}^k \to \mathbb{R}$ de $\mathcal{L}$ est une application $C^2$ régulière qui permet d'exprimer la sous-variété

Lagrangienne comme

$$\mathcal{L} = \{(q, \partial_q S(q,\xi))\ ;\ \partial_\xi S(q,\xi) = 0\}$$

On demande aussi que $\mathcal{L}$ puisse être vue comme une sous-variété de $M \times \mathbb{R}^k$. Plus précisément, on demande

1. premièrement, que 0 soit une valeur régulière de l'application $(q,\xi) \longmapsto \partial_\xi S(q,\xi)$ de manière à ce que l'ensemble critique

$$\Sigma_S = \{(q,\xi)\ ;\ \partial_\xi S(q,\xi) = 0\}$$

sois une sous-variété de $M \times \mathbb{R}^k$.

2. deuxièmement, que l'application

$$\begin{array}{cccc} i_S : & \Sigma_S & \longrightarrow & T^*M \\ & (q,\xi) & \longmapsto & (q, \partial_q S(q,\xi)) \end{array}$$

soit un difféomorphisme de $\Sigma_S$ sur la sous-variété Lagrangienne $\mathcal{L}$.

Une manière de voir ces fonction génératrices est de se dire qu'on ajoute des variables $\xi \in \mathbb{R}^k$ pour pouvoir voire $\mathcal{L}$ comme un graphe partiel au-dessus de $M \times \mathbb{R}^k$. En particulier, si $\mathcal{L}$ est le graphe d'une forme exacte $du$ dans $T^*M$, alors $u : M \to \mathbb{R}$ est une fonction génératrice de $\mathcal{L}$.

Cependant, l'existence de telles fonctions génératrices n'est pas toujours garantie. Un calcul montre que l'application $h = S \circ \imath_S^{-1} : \mathcal{L} \to \mathbb{R}$ vérifie $dh = \lambda_{|\mathcal{L}}$, où $\lambda = pdq$ est la forme de Liouville définie de sorte que $\omega = -d\lambda$. Il en résulte que la restriction de la forme de Liouville $\lambda$ à $\mathcal{L}$ est une forme exacte dont $h$ est une primitive. On dit que $\mathcal{L}$ est une sous-variété Lagrangienne exacte. On observe ici que c'est une condition nécessaire pour l'existence d'une fonction génératrice. Il reste cependant inconnu si elle est suffisante.

Un résultat par J.-C. Sikorav [Sik86, Sik87] énonce l'existence de fonctions génératrices $S$ pour les sous-variétés Lagrangiennes $\mathcal{L}$ Hamiltoniennement isotopes, ou H-isotopes, à la section nulle $0_{T^*M}$, c'est-à-dire les $\mathcal{L} = \varphi^1(0_{T^*M})$ où $\varphi^t$ est un flot Hamiltonien (possiblement non-autonome). De plus, les fonctions génératrices $S$ obtenues peuvent être choisies quadratiques à l'infini, c'est à dire qu'à l'extérieur d'un compact de $M \times \mathbb{R}^k$, $S(q,\xi) = Q(\xi) + c$ pour une forme quadratique non-dégénérée $Q$ et une constante réelle $c \in \mathbb{R}$. On note f.g.q.i pour "fonction génératrice quadratique à l'infini".

Suivant des idées de Marc Chaperon [Cha91], Alberto Ottolenghi et Claude Viterbo [OV95] construisent à partir de ces f.g.q.i et pour toute condition initiale $C^1$-régulière $u_0$, une application $u(t,q) : [0,T] \times M \to \mathbb{R}$ globalement Lipschitzienne, $C^1$-régulière sur un ensemble ouvert dense, qui est est solution de l'équation de Hamilton-Jacobi non-stationnaire

$$\partial_t u + H(q, d_q u) = \alpha$$

Afin de construire ces solutions, Viterbo construit une famille régulière de f.g.q.i $S_t$ associées aux sous-variétés Lagrangiennes $\mathcal{L}_t := \phi_H^t(\mathcal{L})$, et dont les primitives de Liouvilles associées $h_t = S_t \circ i_{S_t}^{-1}$ vérifient pour tout point $(q,p) \in \mathcal{L}$ d'orbite $(q(t), p(t)) = \phi_H^t(q,p) \in \mathcal{L}_t$,

$$h_t(q(t), \xi) := h_0(q, \xi) + \int_0^t \Big( p(\tau).\dot{q}(\tau) - H(q(\tau), p(\tau)) \Big)\, d\tau + \alpha.t \qquad (2.3.1)$$

Ensuite, il associe à chacune de ces f.g.q.i $S_t$ de $\mathcal{L}_t$ un *sélecteur de graphe* $u(t, \cdot)$ dont nous détaillons l'idée et la définition dans le reste de cette section.

L'indice $m$ d'une f.g.q.i $S$ est la dimension de l'espace vectoriel maximal $E^-$ où la forme quadratique $Q$ est définie négative. Fixons un point $q$ de $M$ et posons $S_q = S_{|\{q\} \times \mathbb{R}^k}$ la restriction de de $S$ à la fibre de $q$ de $M \times \mathbb{R}^q$.

Pour tout réel $a \in \mathbb{R}$, on considère le sous-niveau $S_q^a$ de $S_q$ défini par

$$S_q^a = \{\xi \in \mathbb{R}^k \mid S_q(\xi) \le a\}$$

Dans ce cas, puisque $S_q - c$ coïncide avec la forme quadratique $Q$, on obtient pour un réel $N > 0$ grand, $S^N = Q^{N-c}$ et $S^{-N} = Q^{-N-c}$. Lorsque $N - c > 0$ et $-N - c < 0$, on note $Q^{N+c} = Q^\infty = S_q^\infty$ et $Q^{-N-c} = Q^{-\infty} = S_q^{-\infty}$. Comme l'illustre la figure 2.4, on obtient que $(S_q^\infty, S_q^{-\infty})$ est homotope à la paire $(\mathbb{D}^m, \partial\mathbb{D}^m)$ où $\mathbb{D}^m$ est le disque de dimension $m$.

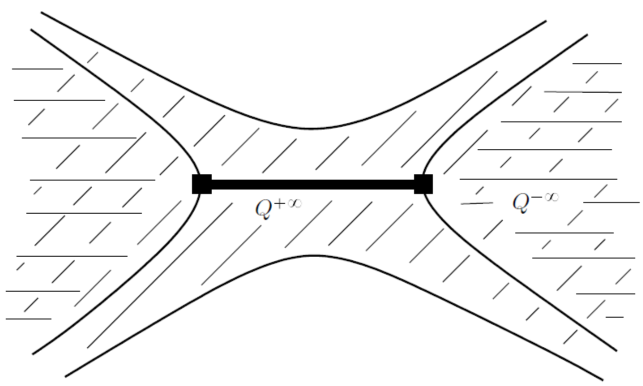



FIGURE 2.4 – Équivalnce d'homotopie $(Q^\infty, Q^{-\infty}) \simeq (\mathbb{D}^m, \partial\mathbb{D}^m)$.
Figure prise de [Hum08].

On en déduit l'équivalence cohomologie (en coefficients réels)

$$H^*(S_q^{\infty}, S_q^{-\infty}) \simeq H^*(\mathbb{D}^m, \partial\mathbb{D}^m) = \begin{cases} \mathbb{R} & \text{si } * = m \\ 0 & \text{si } * \neq 0 \end{cases}$$

générée par un élément $\alpha_q$ d'indice $m$. De plus, Les inclusions $S_q^a \to S_q^{\infty}$ induisent des morphismes cohomologiques

$$\begin{array}{ccc} H^*(S_q^{\infty}, S_q^{-\infty}) & \longrightarrow & H^*(S_q^a, S_q^{-\infty}) \\ \alpha_q & \longmapsto & \alpha_q \end{array}$$

qui permettent de voir la classe $\alpha_q$ comme un élément de $H^*(S_q^a, S_q^{-\infty})$. Mais puisque $H^*(S_q^{-\infty}, S_q^{-\infty}) = 0$, il en résulte que cette classe $\alpha_q$ s'annule dans $H^*(S_q^{-\infty}, S_q^{-\infty}) = 0$. On peut alors s'intéresser à la persistance de $\alpha_q$ initialement non-nulle et considérer l'application

$$u(q) = \inf\{a \in \mathbb{R} \ ; \ \alpha_q \neq 0 \text{ in } H^*(S_q^a, S_q^{-\infty})\}$$

Il s'avère que $u(q)$ est une valeur critique de $S_q$, marquant un changement homotopique en ce niveau d'énergie. Et une étude plus poussée de cette application révèle que celle-ci est globalement Lipschitzienne et est $C^1$-régulière sur un ouvert dense $U$ de $M$ où elle vérifie

$$(q, du(q)) \in \mathcal{L} \quad \text{et} \quad h(q, du(q)) = u(q) \tag{2.3.2}$$

où $h = S \circ i_S^{-1}$ est la primitive de Liouville sur $\mathcal{L}$ associée à sa f.g..i $S$. En particulier, le graphe de $du$ au-dessus de l'ouvert dense $U$ est inclus dans la sous-variété Lagrangienne $\mathcal{L}$. On dit que $u : M \to \mathbb{R}$ est un *sélecteur de graphe* de $\mathcal{L}$ associé à la f.g.q.i $S$. Ceci est illustré dans la figure 2.5.

Par ailleurs, un théorème d'unicité des sélecteurs de graphes [Thé99, AV17] énonce que le sélecteur de graphe d'une sous-variété Lagrangienne $\mathcal{L}$ est unique à une constante près.

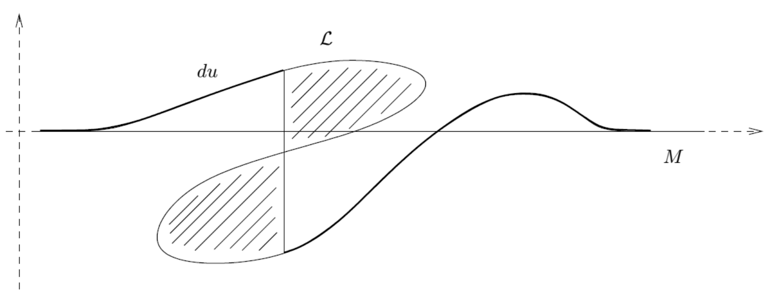



FIGURE 2.5 – Sélecteur de graphe $du$ de $\mathcal{L}$ représenté en gras.
La discontinuité du graphe de $du$ survient lorsque les deux surfaces hachurées sont égales.
Figure prise de [Hum08].

De retour aux f.g.q.i $S_t$ construites par Viterbo, les propriétés des sélecteurs de graphes $u(t,\cdot)$ dans son ensemble ouvert dense s'écrivent

$$(q, d_q u(t,q)) \in \mathcal{L}_t \quad \text{et} \quad h_t(q, d_q u(t,q)) = u(t,q)$$

L'étude de la seconde équation, additionnée à l'identité (2.3.1) donnent exactement l'équation de Hamilton-Jacobi non-stationnaire (2.1.8). Ils montrent enfin par des arguments de continuité en topologie Lipschitz que ces résultats se prolongent pour des conditions initiales lipschitziennes, et en particulier pour les sélecteurs de graphes de sous-variétés Lagrangiennes.

Dans le cas d'un Hamiltonien de Tonelli, lorsque la condition initiale $u_0$ est semi-convexe, il est démontré que la notion de solution variationnelle coïncide avec celle de solution de viscosité générée par le semi-groupe de Lax-Oleinik. Cela permet d'appliquer les résultats de la théorie KAM-faible, notamment ceux concernant l'existence des courbes calibrées ainsi que la différentiabilité et l'unicité sur les ensembles de Mather et d'Aubry, dont la validité n'était pas claire par la définition variationnelle.

La figure 2.6 représente dans le cas du pendule la solution variationnelle de condition initiale $u_0 = 0$. Puisqu'elle coïncide avec la solution de viscosité $\mathcal{T}^t 0(x)$, nous constatons, comme le prédit le théorème de convergence de Fathi 2.1.2, qu'elle converge vers une solution KAM-faible $u_-$. Mieux encore, le graphe de la différentielle $d_q u(t,\cdot)$ converge en topologie $C^0$ (pour la distance de Hausdorff), vers le graphe de la différentielle de $du_-$. Cette convergence est connue et a été démontrée par M-.C. Arnaud dans [Arn05].

### 2.3.2 Théorème de Birkhoff multidimensionnel

En utilisant les solutions variationnelles, M-.C. Arnaud [Arn10] a montré le théorème de Birkhoff multidimensionnel suivant.

**Théorème 2.3.1.** *Soit $H : T^*M \to \mathbb{R}$ un Hamiltonien de Tonelli sur une variété compacte connexe $M$. Soit $\mathcal{L}$ une sous-variété Lagrangienne Hamiltoniennement isotope à la section nulle de $T^*M$. Si $\mathcal{L}$ est invariante par le flot Hamiltonien $\phi_H$, alors c'est un graphe Lipschitz au-dessus de la section nulle.*

La construction de Viterbo donne pour un sélecteur de graphe $u_0$ de la sous-variété Lagrangienne invariante $\mathcal{L}$, une solution variationnelle $u(t,q)$ où pour tout temps $t > 0$, $u(t,\cdot)$ est aussi un sélecteur de graphe de $\mathcal{L}_t = \phi_H^t(\mathcal{L}) = \mathcal{L}$, invariant par le flot Hamiltonien. De plus, le théorème d'unicité à constante près des sélecteurs de graphe donne pour tout temps $t > 0$, une constante $c_t$ telle que $u(t,\cdot) = u_0 + c_t$.

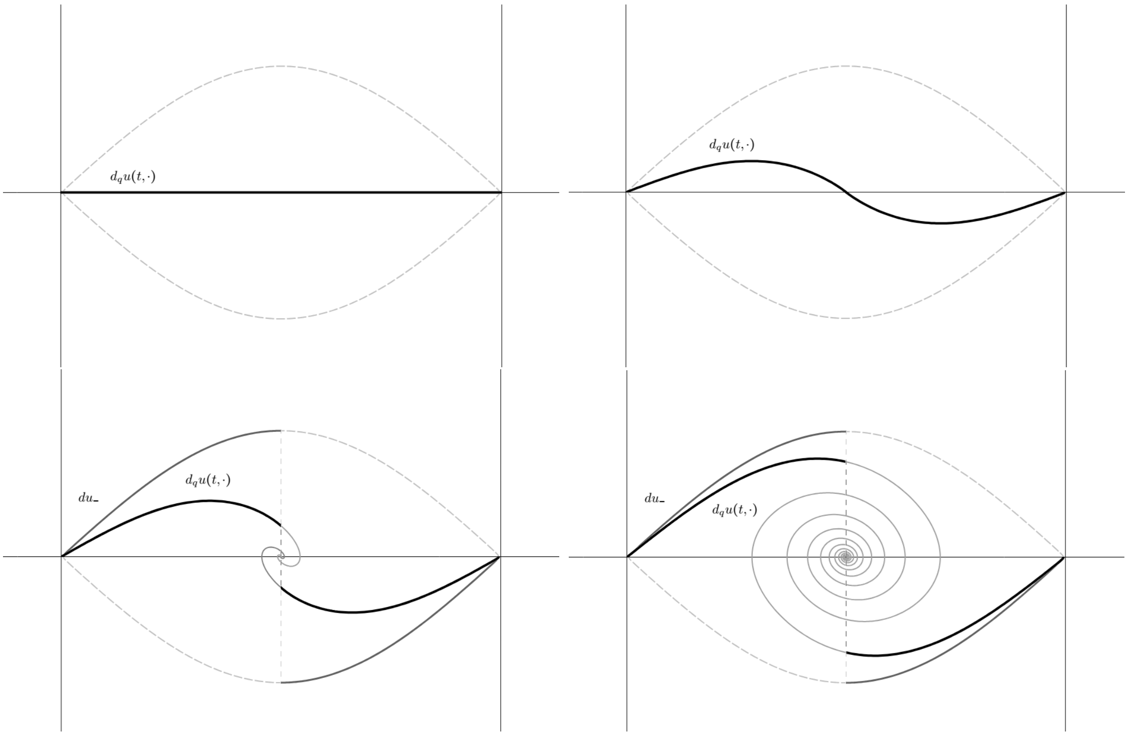



FIGURE 2.6 – Convergence d'une solution variationnelle $u(t,q)$ vers une solution KAM-faible $u_-$ du pendule simple.

Cependant, il n'est pas établi que que la condition initiale $u_0$ soit nécessairement semi-concave. Il est donc impossible de confirmer directement que celle-ci est également une solution de viscosité de l'équation de Hamilton-Jacobi non-stationnaire.

L'objectif est toutefois de démontrer que $\mathcal{L}$ est un graphe au-dessus de la section nulle, et donc qu'il s'agit du graphe continu de la différentielle de son sélecteur de graphe $du(t,\cdot) = du_0$, ce qui implique que nous avons bien une solution de viscosité. Il en découle que $c_t$ converge vers 0, et que $u_0$ est solution KAM-faible de

$$H(q, d_q u_0) = \alpha_0$$

On constate en particulier que $\mathcal{L} \subset \{H = \alpha_0\}$ avec $\alpha_0$ la valeur critique de Mañé . L'application $u_0$ vérifie donc les propriétés des solutions KAM-faibles, notamment, son ensemble d'Aubry relatif projeté $\mathcal{A}_{u_0}$, défini dans (2.1.10), est égal à la variété $M$ tout entière.

L'idée de d'Arnaud a été de prendre le problème à l'envers et de montrer directement que pour tout sélecteur de graphe $u_0$ de $\mathcal{L}$, on doit avoir $\mathcal{A}_{u_0} = M$.

Pour ce faire, elle considère un point $q$ de différentiabilité du sélecteur de graphe $u_0$.

Ce point vérifie $x := (q, d_q u_0) \in \mathcal{L}$. Et elle s'intéresse à la courbe $x(t) = \phi_H^t(x)$, et particulièrement à ses points limites.

Elle montre grâce à l'invariance de $\mathcal{L}$ et à son appartenance au niveau critique $\{H = \alpha_0\}$ que l'ensemble non-errant de $\phi_{H|\mathcal{L}}^t$ est inclus dans l'ensemble d'Aubry $\tilde{\mathcal{A}}^*$. En particulier, les ensembles $\alpha$- et $\omega$-limites $\alpha(x(t))$ et $\omega(x(t))$ sont inclus dans $\tilde{\mathcal{A}}^*$.

Ensuite, en prenant deux points $x_\alpha = (q_\alpha, p_\alpha) \in \alpha(x(t))$ et $x_\omega = (q_\omega, p_\omega) \in \omega(x(t))$, elle évalue, en utilisant (2.3.1), les quantités $u(q(t)) - u(q)$ et passe à la limite pour obtenir

$$\begin{cases} u(q) = u(q_\alpha) + h^\infty(q_\alpha, q) \\ u(q) = u(q_\omega) - h^\infty(q, q_\omega) \end{cases}$$

où $h^\infty$ est la barrière de Peierls et où $q_\alpha$ et $q_\omega$ sont dans l'ensemble d'Aubry.

La première identité rappelle la formule de représentation (2.2.5) spécifique aux solutions KAM-faibles. Cette équation permet de construire une courbe $\gamma_q^- : (-\infty, 0] \to M$ arrivant en $\gamma_q^-(0) = q$, et qui soit $u_0$-calibrée. Mais la deuxième identité, tout à fait analogue, permet de construire une nouvelle courbe $\gamma_q^+ : [0, +\infty) \to M$ arrivant en $\gamma_q^+(0) = q$ et qui soit aussi $u_0$-calibrée. La concaténation de ces deux courbes résulte en une nouvelle $\gamma_q : \mathbb{R} \to M$ qui reste toujours $u_0$-calibrée. Une telle courbe est semi-statique et globale, et appartient donc à l'ensemble d'Aubry projeté relatif $\mathcal{A}_{u_0}$ de $u_0$. En particulier, $q \in \mathcal{A}_{u_0}$.

Comme l'ensemble de différentiabilité de l'application lipschitzienne $u_0$ est dense dans $M$, on en déduit que $\mathcal{A}_{u_0}$ est dense dans $M$. Et puisque c'est un ensemble fermé, nous en déduisons que $\mathcal{A}_{u_0} = M$ et que $u_0$ est différentiable sur $M$ tout entier. Une étude plus poussé de la régularité des solutions de viscosité sur les courbes calibrées donne la régularité $C^{1,1}$ souhaitée de $u_0$.

□

La validité de ce théorème ne s'arrête pas seulement aux sous-variétés Lagrangiennes exactes, mais il peut s'étendre à d'autres niveaux d'énergie en considérant le Hamiltonien translaté

$$H_c = H(q, p + c)$$

À titre d'exemple, regardons ce qui se passe pour le pendule simple $H : (q, p) \in T^*\mathbb{T}^1 \to \frac{1}{2}p^2 + cos(2\pi q) \in \mathbb{R}$. Lorsque $|c| > 2h^\infty(0, 1/2)$, aire de la partie délimitée par le niveau critique $\{H = 1\}$ et l'axe des abscisses, on obtient que les courbes invariantes représentées dans 2.7 sont exactes. Le théorème stipule alors qu'elles forment des graphes lipschitziens au-dessus de $\mathbb{T}^1$, ce qui est confirmé graphiquement.

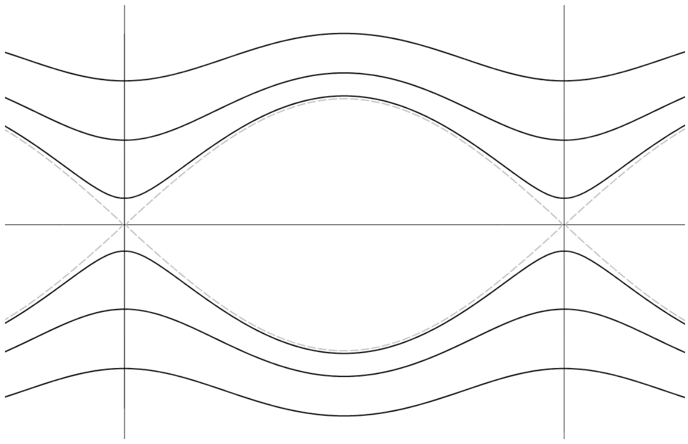


FIGURE 2.7 – Courbes essentielles invariantes par le pendule simple.

Par ailleurs, ce théorème s'étend aux parties supérieures et inférieures du niveau critique $\{H = 1\}$ qui sont aussi des graphes Lipschitz au-dessus de $\mathbb{T}^1$. En effet, Bernard et Dos Santos [BdS12] ont étendu le résultat pour les Lagrangiennes exactes-Lipschitz $\mathcal{L}$ qui sont images $\mathcal{L} = \varphi(du)$ par le temps 1 d'un flot Hamiltonien $\phi$ d'une forme exacte lipschitzienne $du$.

Ce théorème offre aussi une nouvelle vision géométrique des ensembles d'Aubry et de Mañé , ainsi que de la valeur critique de Mañé $\alpha_0$. Dans leurs travaux, Bernard et Dos Santos [BDS10, BdS12] dérivent ces ensembles et valeurs en terme de sous-variétés Lagrangiennes. En considérant les ensembles

$$\mathcal{I}_{\alpha_0}(\mathcal{L}) = \bigcap_{t\in\mathbb{R}} \phi_H^t(\mathcal{L} \cap \{H = \alpha_0\})$$

ils démontrent que

$$\alpha_0 = \inf_{\mathcal{L}} \max_{(q,p)\in\mathcal{L}} H(q,p)$$

et

$$\tilde{\mathcal{A}}^* = \bigcap_{\mathcal{L}\subset\{H\leq\alpha_0\}} \mathcal{I}_{\alpha_0}(\mathcal{L}) \quad , \quad \tilde{\mathcal{N}}^* = \bigcup_{\mathcal{L}\subset\{H\leq\alpha_0\}} \mathcal{I}_{\alpha_0}(\mathcal{L})$$

où les sous-variété Lagrangiennes $\mathcal{L}$ décrivent les sous-variétés Hamiltoniennement isotopes à la section nulles. Le théorème de Birkhoff 2.3.1 indique qu'il suffit de se restreindre aux Lagrangiennes qui sont des graphes des 1-forme exactes Lipschitz $du$.

Ce théorème illustre parfaitement la puissance des outils de la théorie KAM faible et l'apport novateur qu'elle offre par rapport aux approches variationnelles et même à l'analyse des viscosité initialement développée par Crandall et Lions. Parmi d'autres applications notables de cette théorie, citons les travaux de Thieullen et Su [ST18] dans le domaine du contrôle optimal, ainsi que l'étude de Maderna et Venturelli [MV20] sur le problème à N-corps.

# Chapitre 3

# Théorie KAM-faible non-autonome

Au-delà du cadre autonome initialement étudié par A. Fathi dans sa théorie KAM-faible, il est possible d'étendre cette théorie au cadre non autonome, où le hamiltonien $H(t, q, p)$ dépend du temps.

On se place toujours sur le fibré cotangent $T^*M$ d'une variété compacte connexe $M$ de dimension $d$. Afin de pouvoir considérer des courbes minimisantes définie sur $\mathbb{R}$ tout entier, nous devons ajouter une nouvelle hypothèse de complétude à la définition 2.0.1 de Tonelli. Cette hypothèse, automatiquement vérifiée dans le cas autonome, est la suivante :

- Complétude : Le champ de vecteurs Hamiltonien $X_H(x, p) = (\partial_p H(x, p), -\partial_x H(x, p))$ et donc son flot $\phi_H$ sont complets dans le sens où les courbes de flot sont définies en tout temps $t \in \mathbb{R}$.

On considère un Hamiltonien $H : \mathbb{T}^1 \times T^*M \to \mathbb{R}$ périodique en temps, et de Tonelli. Si l'on note $H_t = H(t, \cdot)$ alors le champs de vecteur Hamiltonien $X_H^t$ est cette fois défini par l'équation

$$\iota_{X_H^t}\omega := \omega(X_H^t, \cdot) = dH_t$$

Et son flot Hamiltonien $\phi_H^{s,t}$ dépend alors de deux variables temporelles $s$ et $t$. Notons que la périodicité du Hamiltonien implique que $\phi_H^{s,t} = \phi_H^{s+k,t+k}$ pour tout entier $k$.

Le Lagrangien de Tonelli $L(t, x, v) : \mathbb{T}^1 \times TM \to \mathbb{R}$ associé est alors donné par la relation

$$L(t, x, v) = \max_{p \in T_x^*M} \{p(v) - H(t, x, p)\} \tag{3.0.1}$$

Et l'application de Legendre $\mathcal{L} : \mathbb{T}^1 \times TM \to \mathbb{T}^1 \times T^*M$ qui conjugue les flots Hamiltonien

$\phi_H^{s,t}$ et Lagrangien $\phi_L^{s,t}$ devient

$$\mathcal{L}(t,x,v) = (t,x,\partial_v L(t,x,v)) \quad \text{ou} \quad \mathcal{L}^{-1}(t,x,p) = (t,x,\partial_p H(t,x,p))$$

Cette nouvelle dépendance temporelle introduit une richesse dynamique supplémentaire, mais entraîne aussi une perte de conservation de l'énergie, une complexité accrue dans les trajectoires, et l'émergence de nouveaux comportements. En effet, l'énergie totale n'est pas conservée puisque le Hamiltonien change avec le temps. Cela complique l'analyse et empêche de réduire la dynamique à des sous-variétés d'énergie fixe. Par conséquent, l'étude de l'équation de Hamilton-Jacobi stationnaire n'a plus de sens dans ce cadre, et toute la théorie KAM-faible non autonome repose alors sur l'étude de l'équation de Hamilton-Jacobi non stationnaire.

$$\partial_t u + H(t, d_x u) = \alpha_0 \tag{3.0.2}$$

Ce cadre d'étude s'inscrit dans la théorie KAM-faible discrète, déjà étudiée dans [Zav12, Zav10, ST18]..., où le potentiel $h_0^t$ est remplacé par une application coût $c : M \times M \to \mathbb{R}$, établissant un lien étroit avec la théorie du transport optimal.

## 3.1 Comparaison avec le cas autonome

**Ce qui ne change pas, ou peu.**

Commençons par pointer le fait que la théorie des mesures minimisantes de Mather a été établie dans un cadre non-autonome pour des mesures définies dans $\mathbb{T}^1 \times TM$. Ainsi, l'ensemble de Mather $\tilde{\mathcal{M}}$ reste toujours bien défini comme la réunion des supports des mesures minimisantes et est inclus dans $\mathbb{T}^1 \times TM$. Par conséquent, la fonction $\alpha : H^1(M,\mathbb{R}) \to \mathbb{R}$ reste toujours définie et le choix de la valeur critique $\alpha_0$ fait sens dans l'équation de Hamilton-Jacobi (3.0.2).

Comme pour les flots non-autonomes, le potentiel $h^{s,t} := h_{\alpha_0}^{s,t}$ et l'opérateur de Lax-Oleinik $\mathcal{T}^{s,t} := \mathcal{T}_{\alpha_0}^{s,t}$ dépendent aussi de deux variables temporelles $s < t$ comme suit

$$h^{s,t}(x,y) = \inf\left\{ A_{L+\alpha_0}(\gamma) = \int_s^t L(\tau,\gamma(\tau),\dot\gamma(\tau))\,d\tau + \alpha_0.(t-s) \;\middle|\; \begin{array}{rl} \gamma: & [s,t] \to M \\ & s \mapsto x \\ & t \mapsto y \end{array} \right\} + \alpha_0.(t-s)$$

et

$$\mathcal{T}^{s,t}u_0(x) = \inf_{\substack{\gamma:[s,t]\to M \\ t\mapsto x}} \left\{ u_0(\gamma(s)) + \int_s^t L(\tau,\gamma(\tau),\dot\gamma(\tau))\,d\tau + \alpha_0.(t-s) \right\}$$

$$= \inf_{y\in M} \left\{ u_0(y) + h^{s,t}(y,x) \right\}$$

Les courbes $\gamma$ sont dorénavant définies dans $[s,t]$ par contraste avec le cas autonome où elles étaient définies sur $[0,t-s]$. On note $\mathcal{T} := \mathcal{T}^{0,1}$. La périodicité en temps du Lagrangien fait que la famille $(\mathcal{T}^n)_{n\in\mathbb{N}}$ est un semi-groupe discret qui vérifie $\mathcal{T}^n = \mathcal{T}^{0,n}$.

Les solutions de viscosité $u(t,x) = \mathcal{T}^{s,t}u_0(x)$ générées par cet opérateur conservent leurs propriétés de régularité mentionnées dans le cas autonome. Elles sont bornées en temps positifs, équi-lipschitziennes. De plus, elles sont dominées dans le sens où

$$u(t,x) - u(s,y) \le h^{s,t}(y,x) \le \int_s^t L(\tau,\gamma(\tau),\dot\gamma(\tau))\,d\tau + \alpha_0.(t-s)$$

pour toute courbe $\gamma:[s,t]\to M$ liant $\gamma(s) = y$ ) $\gamma(t) = x$. Et si $u$ est défnie sur $\mathbb{R}$, alors pour tout point $(t,x)$ de $\mathbb{R}\times M$, on peut trouver une courbe $\gamma_x : (-\infty,t]$ arrivant en $\gamma(t) = x$ qui soit $u$-calibrée, i.e. telle que pour tout temps $s<t$,

$$u(t,x) - u(s,\gamma_x(s)) = h^{s,t}(\gamma_x(s),x) = \int_s^t L(\tau,\gamma_x(\tau),\dot\gamma_x(\tau))\,d\tau + \alpha_0.(t-s)$$

Ces solutions de viscosité conservent leurs différentiabilités sur les courbes calibrées et le graphe de leurs différentielle demeure invariant, en temps négatifs, par le flot hamiltonien.

Une nouveauté dans la définition du potentiel de Mañé $m$ et de la barrière de Peierls $h^\infty$ est qu'elles nécessitent de prendre des temps entiers et non réels. Nous définissons ces quantités par

$$m^{s,t}(x,y) = \inf_{n>t-s} h^{s,t+n}(x,y) \quad \text{et} \quad h^{s,\infty+t} = \liminf_{n} h^{s,t+n}(x,y)$$

Il est à noter que la barrière de Peierls $h^\infty(t,x,y) := h^{0,\infty+t}(x,y)$ est 1-périodique en temps. Ces quantités permettent alors de définir les courbes semi-statiques et statiques $\gamma$, qui à leur tour définissent les ensembles d'Aubry $\tilde{\mathcal{A}}$ et de Mañé $\tilde{\mathcal{N}}$ dans $\mathbb{T}^1\times TM$ comme la réunion de $(t,\gamma(t),\dot\gamma(t))$ pour les courbes globales correspondantes. On définit en particulier le temps 0 de ces ensembles comme

$$\begin{array}{rcl} \tilde{\mathcal{M}}_0 = \tilde{\mathcal{M}} \cap \big(\{0\} \times TM\big) & , & \mathcal{M}_0 = \pi(\tilde{\mathcal{M}}_0) = \mathcal{M} \cap \big(\{0\} \times M\big) \\ \tilde{\mathcal{A}}_0 = \tilde{\mathcal{A}} \cap \big(\{0\} \times TM\big) & , & \mathcal{A}_0 = \pi(\tilde{\mathcal{A}}_0) = \mathcal{A} \cap \big(\{0\} \times M\big) \\ \tilde{\mathcal{N}}_0 = \tilde{\mathcal{M}} \cap \big(\{0\} \times TM\big) & & \end{array} \tag{3.1.1}$$

La dynamique entre ces ensembles reste inchangée par rapport au cas autonome, à l'exception de devoir prendre des temps entiers dans les définitions des $\alpha$ et $\omega$-limites des courbes.

Relevons, enfin, que la théorie des solutions variationnelles également été développée dans le cas non-autonome par Viterbo et Ottolenghi. De ce fait, les idées de base de la preuve du théorème de Birkhoff multidimensionnel restent valables dans un cas non-autonome. Ceci a permis une généralisation de ce résultat par Arnaud et Venturelli [AV17] pour les sous-variétés Lagrangiennes invariantes par le temps 1 d'un flot Hamiltonien non-autonome et de Tonelli $\phi^1_H$.

**Ce qui change.**

Une nouveauté notable du cas non-autonome est que le théorème de convergence de Fathi 2.1.2 n'est plus vrai. En effet, il donne avec Mather [FM00] un exemple de Hamiltonien de Tonelli et de solution de viscosité $u(t,x) = \mathcal{T}^t u_0(x)$ qui ne converge pas.

Pire que cela, il existe même des Hamiltoniens de Tonelli non-autonomes pour lesquelles l'ensemble $\bigcap_{t>0} \mathrm{Fix}(\mathcal{T}^t)$ est vide. La définition de solutions KAM-faibles stationnaires ne tient alors plus, et il devient légitime de s'intéresser au comportement asymptotique du semi-groupe de Lax-Oleinik $(\mathcal{T}^n)_{n>0}$ que l'on sait faiblement contractant et borné. On définit alors

1. L'ensemble de ses points fixes $\mathrm{Fix}(\mathcal{T}) = \{u \in \mathcal{C}(M,\mathbb{R}) \ ; \ \mathcal{T}u = u\}$. Les éléments de $\mathrm{Fix}(\mathcal{T})$ sont appelés *solutions KAM-faibles* de l'équation de Hamilton-Jacobi non-stationnaire (3.0.2).
2. L'ensemble de ses points périodiques $\mathrm{Per}(\mathcal{T})$ formé des applications scalaires $u \in \mathcal{C}(M,\mathbb{R})$ telles que $\mathcal{T}^n u = u$ pour un entier $n \geq 1$. En particulier, pour tout entier $N \geq 1$ fixé, l'ensemble des points $N$-périodiques $\mathcal{T}^N u = u$ est noté $\mathrm{Per}_N(\mathcal{T})$.
3. L'ensemble de ses points récurrents $\mathcal{R}(\mathcal{T})$ formé des applications scalaires $u \in \mathcal{C}(M,\mathbb{R})$ telles que $\mathcal{T}^{k_n} u$ converge vers $u$ pour une certaine suite strictement croissante d'entier $k_n \geq 0$. Ceci peut-être reformulé en $u \in \omega(u)$ où

$$\omega(u) = \{v \in \mathcal{C}(M,\mathbb{R}) \mid \exists (k_n)_n \in \mathbb{N}^{\mathbb{N}} \text{ suite strictement croissante telle que} \|\mathcal{T}^{k_n} u - v\|_\infty \to 0 \text{ as } n \to \infty\}$$

4. Son ensemble non-errant $\Omega(\mathcal{T})$ formé des applications scalaires $u \in \mathcal{C}(M,\mathbb{R})$ telles que tout voisinage $U$ de $u$ dans $\mathcal{C}(M,\mathbb{R})$ vérifie $\mathcal{T}^n U \cap U \neq \varnothing$ pour un certain entier $n > 1$.

Il est essentiel de noter la distinction entre solutions de viscosité et solutions KAM-faibles qui ne coïncident plus dans le cadre non-autonome. Les solutions de viscosité sont toutes les solutions générées par l'opérateur de Lax-Oleinik $\mathcal{T}^{s,t}$ alors que les solutions KAM-faibles sont les points fixes de son temps 1 $\mathcal{T}^{0,1}$. En d'autres termes, ce sont les solutions de viscosité 1-périodiques en temps.

Comme mentionné après la définition des barrières de Peierls $h^\infty$, celle-ci offrent une famille de solutions KAM-faibles $h^\infty(x,\cdot)$ qui permettent la description générale de tout $\mathrm{Fix}(\mathcal{T})$. Il en résulte qu'une formule de représentation reste toujours valable dans le cas non-autonome, comme prouvé par G. Contreras, R. Iturriaga, et H. Sánchez-Morgado dans [CISM13].

Tournons notre attention vers le nouvel ensemble non-errant $\Omega(\mathcal{T})$. Celui-ci continent les points limites de toutes les solutions de viscosité. Il encode alors toute l'information du comportement asymptotique du semi-groupe de Lax-Oleinik $\mathcal{T}$. Cet ensemble représente le centre d'intérêt principal de cette thèse, dans laquelle nous avons cherché à en comprendre la structure et les propriétés.

Une première observation est que la faible contractivité de $\mathcal{T}$ implique que les notions de non-errance et de récurrence sont équivalentes pour $\mathcal{T}$, i.e. $\Omega(\mathcal{T}) = \mathcal{R}(\mathcal{T})$.

Un théorème par P. Bernard et J-.M. Roquejoffre [BR04] montre qu'en dimension 1, cet ensemble non-errant $\omega(\mathcal{T})$ coïncide aussi avec l'ensemble des solutions de viscosité périodiques $\mathrm{Per}(\mathcal{T})$. Ils prouvent alors par la même occasion que l'exemple construit par Fathi et Mather était aussi périodique.

Pour montrer la distinction entre ces ensembles, nous avons réussi à construire un exemple de solution de viscosité récurrente et non périodique en dimension supérieure ou égale à 2. Mais avant de plonger dans les résultats obtenus, nous proposons d'examiner les différences entre le cas autonome et le cas non-autonome sur l'exemple du pendule.

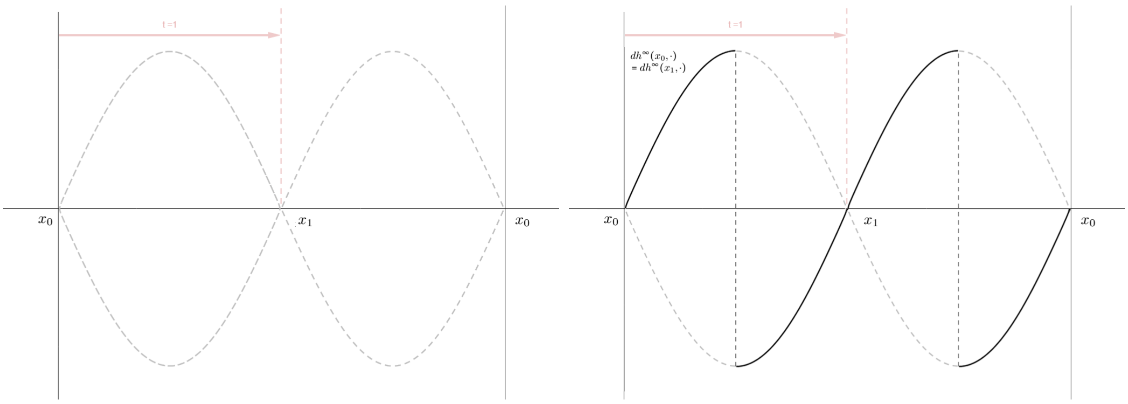
## 3.2 Non convergence du semi-groupe de Lax-Oleinik associé au 2-pendule non-autonome

Cette étude a débuté par l'analyse de l'exemple de Fathi et Mather [FM00] d'un Hamiltonien non-autonome et de Tonelli dont le semi-groupe de Lax-Oleinik $\mathcal{T}$ ne converge pas.

Nous observons que leur exemple, plus abstrait, peut être illustré par le cas du 2-pendule translaté dans le temps, dont le Hamiltonien $H : \mathbb{T}^1 \times T^*\mathbb{T}^1 \to \mathbb{R}$ est donné par

$$H(t,x,p) = \frac{1}{2}p^2 + \cos\left(4\pi\left(x - \frac{t}{2}\right)\right)$$

Ce Hamiltonien possède le même portrait de phase que le 2-pendule autonome, à la différence qu'il se déplace linéairement dans le temps de sorte qu'à $t = 1$, le point $x_0$ est envoyé sur le point $x_1$, comme l'indique la figure 3.1a.




(a) Niveau d'énergie critique.

(b) Solution KAM-faible.

FIGURE 3.1 – Portrait de phase du 2-pendule non-autonome.

Identifions les solutions KAM-faibles $u_-$ de ce système. En suivant la démarche du cas autonome présentée dans la sous-section 2.2.3, on trouve une famille à un paramètres $du_c$ de solutions possibles, représentées dans la figure 2.3b. En tenant compte de la translation temporelle propre au cas non autonome et de la nouvelle définition des solutions KAM-faibles comme points fixes de l'opérateur $\mathcal{T}^{0,1}$, il est nécessaire que la solution KAM-faible $u_-$, ainsi que sa différentielle $du_-$, soient 1-périodique en temps. Dans cette famille de candidats, la seule solution possible est celle dont les deux points de discontinuité $x_c$ et $1 - x_c$ s'envoient l'un sur l'autre par translation temporelle $t = 1$. Cette solution est représentée dans la figure 3.1b.

L'unicité des différentielles pour les solutions KAM-faibles s'explique par la formule de représentation (2.2.5) valable dans le cas non-autonome. En effet, les points $x_0$ et $x_1$ appartienne désormais à la même classe statique car $h^{2n+1}(x_0,x_1) = h^{2n+1}(x_1,x_0) = 0$ sont annulés par les courbes $\gamma_0(t) = t \in \mathbb{T}^1$ et $\gamma_2(t) = 1/2 + t \in \mathbb{T}^1$, ce qui résulte en

$$h^\infty(x_0,x_1) = h^\infty(x_1,x_0) = 0 \quad \text{et} \quad d(x_0,x_1) = h^\infty(x_0,x_1) + h^\infty(x_1,x_0) = 0$$

Ceci donne $\mathbb{M} = \{x_0\}$ et $dh^\infty(x_0, \cdot) = dh^\infty(x_1, \cdot)$ est la seule différentielle possible pour les solutions KAM-faibles.

Dans le cas autonome, les solutions de viscosité convergeaient vers la famille à deux paramètres $u_{c,c'}$ de solutions KAM-faibles. Cependant, puisque la grande majorité de ces solutions ne sont plus KAM-faibles dans le cas non autonome, il est naturel de s'attendre à ce que cette convergence ne soit plus valable. Il est donc pertinent de s'intéresser à ce que deviennent les solutions disparues : elles deviennent des solutions de viscosité 2-périodiques $u \in \mathrm{Per}_2(\mathcal{T})$.

Dans leur étude, Fathi et Mather introduisent une généralisation des barrières de Peierls, notée $h^{n\infty}$ et définies pour tous entier $n \geq 1$ et temps positif $t > 0$ par

$$h^{n\infty+t}(x,y) = \liminf_i h^{ni+t}(x,y) \tag{3.2.1}$$

où $h^t = h^{0,t}$. Ces barrières présentent peu d'intérêt dans la cas autonome à cause de la convergence du potentiel $h^t$ vers la barrière de Peierls $h^\infty$ exprimée dans (2.2.2). Mais dans le cas non-autonome, elle présente un élément majeur pour la compréhension du comportement asymptotique des solutions de viscosité.

Et en considérant l'ensemble d'Aubry projeté en temps $t = 0$, $\mathcal{A}_0 = \mathcal{A} \cap \{t = 0\}$, Ils définissent ensuite l'application $d_n : \mathcal{A}_0 \times \mathcal{A}_0 \to \mathbb{R}$ donnée par

$$d_n(x,y) = h^{n\infty}(x,y) + h^{n\infty}(y,x)$$

Cette application induit une relation d'équivalence $x \simeq_n y \leftrightarrow d_n(x,y) = 0$ dont les classes d'équivalence $\mathbb{A}_n$ généralisent les classes statiques en sous-classes $n$-statiques pouvant être représentés par des éléments $\mathbb{M}_n$ de l'ensemble de Mather $\mathcal{M}_0 = \mathcal{M} \cap \{t = 0\}$.

Fathi et Mather expriment alors une condition suffisante pour la non-convergence du semi-groupe de Lax-Oleinik $\mathcal{T}$ qui s'exprime comme suit :

**Théorème 3.2.1.** *Si pour un entier $n \geq 2$, il existe deux points $x$ et $y$ de $\mathcal{A}_0$ ayant la même classe statique dans $\mathbb{A}$, mais appartenant à des $n$-classes statiques distinctes dans $\mathbb{A}_n$, alors il existe une condition initiale $u \in \mathcal{C}(M, \mathbb{R})$ dont l'orbite sous l'action du semi-groupe de Lax-Oleinik $(\mathcal{T}^n u)$ ne converge pas.*

En effet, pour tout point $x$ de $M$, les applications $h^{n\infty+t}(x, \cdot)$ sont des solutions de viscosité $n$-périodiques de l'équation de Hamilton-Jacobi (3.0.2). Ainsi, si

$$0 \neq d(x,y) = h^\infty(x,y) + h^\infty(y,x) \leq d_n(x,y) = h^{n\infty}(x,y) + h^{n\infty}(y,x) \neq 0$$

alors l'une des inégalités $h^\infty \leq h^{n\infty}$ est stricte et, par exemple, $h^{n\infty}(x,y) \neq h^\infty(x,y)$. Par conséquent, $h^{n\infty}(x,\cdot)$ n'est pas 1-périodique, et son orbite $\{h^{n\infty+k}(x,\cdot)\}_k$ sous l'action du semi-groupe de Lax-Oleinik $\mathcal{T}$ possède au moins deux éléments distincts, ce qui empêche la convergence.

L'analyse de ce résultat par Fathi et Mather est présentée plus en détail dans l'appendice A. Nous y proposons une réinterprétation de leur formule pour $d_n(x,y)$ en termes d'aires, applicable dans un cas légèrement plus général que celui du pendule.

Retournons à l'application au 2-pendule non-autonome. Notre but est de regarder les solutions de viscosité 2-périodiques. Il alors naturel de prendre $n = 2$ pour les objets introduits plus haut.

Constatons qu'une dilatation temporelle du Hamiltonien, avec $H_2 = 2H(2t,x,p)$, transforme les solutions 2-périodiques pour $H$ en solutions KAM-faibles pour $H_2$. Une formule de représentation (2.2.5) est alors valable en replaçant $\mathbb{M}$ par $\mathbb{M}_2$ et $h^\infty$ par $h^{2\infty}$. On obtient ainsi une famille à deux paramètres $u_{c_0,c_1}$ solutions de viscosité 2-périodiques données par

$$u_{c_0,c_1}(x) = \inf\left\{c_0 + h^{2\infty}(x_0,x), c_1 + h^{2\infty}(x_1,x)\right\}$$

et les barrières $h^{2\infty}(x_i,\cdot)$ sont données par la formule

$$h^{2\infty}(x_i,y) = \sup_{u\in \mathrm{Per}_2(\mathcal{T})}\{u(y)-u(x_i)\} = \sup_{\substack{u\in \mathrm{Per}_2(\mathcal{T})\\ u(x_i)=0}}\{u(y)\}$$

Les graphes des différentielles de ces solutions sont analogues à ceux de la figure 2.3c, avec $h^\infty$ remplacé par $h^{2\infty}$ et en tenant compte de la translation temporelle.

Les solutions KAM-faibles du cas autonomes se sont donc transformée en solutions de viscosité 2-périodiques du pendule non-autonome. Il est alors légitime de se dire que les solutions de viscosité dans notre cas vont converger vers les éléments de $\mathrm{Per}_2(\mathcal{T})$. Ceci s'avère vrai comme prouvé par P. Bernard et J.-M. Roquejoffre dans [BR04] qui ont montré que dans le cas du cercle $M = \mathbb{T}^1$, il existe un entier $N \geq 1$ tel que toute solution de viscosité converge vers un élément de $\mathrm{Per}_N(\mathcal{T})$. Cet entier est égal à 2 dans notre cas.

Dans ce contexte, l'ensemble pertinent devient celui des solutions de viscosité 2-périodiques $\mathrm{Per}_2(\mathcal{T})$ plutôt que l'ensemble des solutions KAM-faibles $\mathrm{Fix}(\mathcal{T})$. En effet, $\mathrm{Per}_2(\mathcal{T})$ généralise les propriétés vérifiées par les solutions KAM-faibles et semble être approprié pour les étendre.

Plus généralement, on peut se demander quelle extension considérer en dimensions supérieures et dans ce cas autonome. Quelles solutions de viscosité conservent les propriétés attendues et vérifiées par les solutions KAM-faibles ? C'est l'objet de cette thèse, dans laquelle nous affirmons que, dans le cadre non-autonome, l'ensemble pertinent à considérer est l'ensemble récurrent $\mathcal{R}(\mathcal{T})$, qui coïncide par faible-contractivité de $\mathcal{T}$ avec l'ensemble non-errant $\Omega(\mathcal{T})$.

# Chapitre 4

# Panorama des résultats de la Thèse

Dans un premier temps, nous démontrons les résultats principaux de la théorie KAM faible dans le cadre non-autonome. Cela s'inscrit dans le cadre plus général de la théorie KAM faible discrète étudiée par Maxime Zavidovique [Zav12, Zav10]. D'autres auteurs ont déjà étendu certains résultats de la théorie KAM faible à ce cadre non-autonome [Ber08, CISM13]. Ensuite, nous nous intéressons à l'ensemble non-errant $\Omega(\mathcal{T})$ du semi-groupe de Lax-Oleinik $\mathcal{T}$ qui semble être beaucoup plus riche que dans le cas autonome.

Dans les chapitres 6 et 7, nous étudions l'action de la restriction de $\mathcal{T}$ à cet ensemble $\Omega(\mathcal{T})$. Une caractérisation des éléments non-errants de $\mathcal{T}$ en tant que solutions de viscosité globales et bornées est énoncée dans le théorème 4.1.1. Puis nous décrivons ces éléments de $\Omega(\mathcal{T})$ par une formule de représentation analogue à ce qui est connu pour l'ensemble $\mathrm{Fix}(\mathcal{T})$ des solutions KAM-faibles (Voir section 2.2.2 et [CISM13]).

Dans le chapitre 8, nous prouvons que l'ensemble $\Omega(\mathcal{T})$ peut contenir une solutions de viscosité récurrente, non-périodique qui de plus est $C^\infty$-régulière. En d'autres termes, nous montrons qu'il est possible de construire, sur toute variété compacte connexe de dimension supérieure ou égale à 2, des Hamiltoniens de Tonelli pour lesquels $\Omega(\mathcal{T}) \smallsetminus \mathrm{Per}(\mathcal{T})$ est non vide. Ceci montre que le résultat de P.Bernard et J.-M. Roquejoffre [BR04] donnant l'égalité $\Omega(\mathcal{T}) = \mathrm{Per}(\mathcal{T})$ en dimension 1 ne se généralise pas aux dimensions supérieures. Cette construction fournit aussi un exemple de sous-variété Lagrangienne régulière $\mathcal{L}$ qui est récurrente en topologie Hausdorff sous l'action du temps 1 d'un flot Hamiltonien Tonelli.

Et dans un dernier chapitre 9, nous prouvons un théorème de Birkhoff (Voir Section 1.2) pour les sous-variétés Lagrangiennes de $T^*M$, récurrentes sous une certaine topologie par un flot Hamiltonien Tonelli. Plus précisément, on suppose que pour une sous-variété Lagrangienne $\mathcal{L}$, Hamiltoniennement isotope à la section nulle, qu'il existe deux suites strictement croissantes d'entiers $m_k$ et $n_k$ de $\mathbb{N}$ telles que les ensemble $\phi_H^{-m_k}(\mathcal{L})$ et $\phi_H^{n_k}(\mathcal{L})$ convergent en distance Hausdorff et avec un contrôle sur leurs longueurs (oscillation de leurs primitives de Liouville) vers deux sous-variétés Lagrangiennes $\mathcal{L}_\alpha$ et $\mathcal{L}_\omega$. On montre alors que $\mathcal{L}$ et toutes ses images $\phi_H^t(\mathcal{L})$ sont des graphes $C^1$ au dessus de la section nulle de $T^*M$, et que de plus, elles sont récurrentes dans le sens où il est possible de choisir $\mathcal{L}_\alpha = \mathcal{L}_\omega = \mathcal{L}$. Ce résultat offre une généralisation du théorème de Birkhoff multidimensionnel démontré par Arnaud et Venturelli (Voir Sections 1.2, 2.3.2 et [Arn10, AV17]).

## 4.1 Action de l'opérateur de Lax-Oleinik sur $\Omega(\mathcal{T})$

Soit $M$ une variété compacte connexe, et soit $H : \mathbb{T}^1 \times T^*M \to \mathbb{R}$ un Hamiltonien Tonelli sur $M$. Il est connu que l'opérateur de Lax-Oleinik $\mathcal{T}$ est faiblement contractant sur l'ensemble des fonctions scalaires $\mathcal{C}(M,\mathbb{R})$, i.e pour tous $u$ et $v$ dans $\mathcal{C}(M,\mathbb{R})$ et tous temps $s < t$

$$\|\mathcal{T}^{s,t}u - \mathcal{T}^{s,t}v\|_\infty \leq \|u - v\|_\infty \tag{4.1.1}$$

Ceci mène à quelques premières implications fondamentales sur son comportement asymptotique, que nous rassemblons dans la proposition suivante. Rappelons que les ensembles particuliers de $\mathcal{T}$ ont été introduits dans la section 3.1.

**Proposition 4.1.1.** *1. Soit $u \in \mathcal{C}(M,\mathbb{R})$. L'ensemble $\omega$-limite $\omega(u)$ de $u$ est compact dans $\mathcal{C}(M,\mathbb{R})$ et la restriction de $\mathcal{T}$ à $\omega(u)$ est minimale, i.e. pour tout $v \in \omega(u)$, $\omega(u) = \overline{\{\mathcal{T}^n v \mid n \in \mathbb{N}\}} = \omega(v)$.*

*2. L'ensemble non-errant $\Omega(\mathcal{T})$ est égal à l'ensemble récurrent $\mathcal{R}(\mathcal{T})$.*

*3. La relation $u \sim v \Leftrightarrow v \in \omega(u)$ est une relation d'équivalence. Si on note $\Lambda$ l'ensemble de ses classes d'équivalence, alors on a*

$$\Omega(\mathcal{T}) = \mathcal{R}(\mathcal{T}) = \bigsqcup_{u \in \Lambda} \omega(u) \tag{4.1.2}$$

*où la réunion est disjointe.*

Ainsi, l'ensemble non-errant de $\mathcal{T}$ coïncide avec son ensemble récurrent et toute solution de viscosité $u(t,\cdot) = \mathcal{T}^t u(0,\cdot)$ a pour points limites des solutions de viscosité récurrentes.

De surcroît, le fait que les ensembles $\omega$-limites $\omega(u)$ soient fermés et $\mathcal{T}$-invariants, on en déduit que la restriction de l'opérateur de Lax-Oleinik $\mathcal{T}$ au compact $\omega(u)$ est 1-Lipschitz et surjective. Il en résulte que cette restriction $\mathcal{T}_{|\omega(u)}$ est une isométrie bijective.

Ceci se généralise à l'ensemble non-errant $\Omega(\mathcal{T})$ et aux opérateur $\mathcal{T}^{s,t}$ comme l'atteste la proposition suivante.

**Proposition 4.1.2.**
1. *La restriction de $\mathcal{T}$ à son ensemble non-errant $\Omega(\mathcal{T})$ est une isométrie bijective, i.e $\mathcal{T}$ est inversible et pour tous $v$ et $w$ dans $\Omega(\mathcal{T})$, $\|\mathcal{T}v - \mathcal{T}w\|_\infty = \|v-w\|_\infty$.*
2. *Plus généralement, si on note $\Omega_\tau(\mathcal{T}) = \mathcal{T}^\tau\big(\Omega(\mathcal{T})\big)$, alors pour tous temps $s < t$, l'opérateur $\mathcal{T}^{s,t} : \Omega_s(\mathcal{T}) \to \Omega_t(\mathcal{T})$ est une isométrie bijective. On note son inverse par $\mathcal{T}^{t,s}$.*
3. *Pour tous temps $s$, $t$ et $\tau$ dans $\mathbb{R}$, $\mathcal{T}^{s,t} = \mathcal{T}^{\tau,t} \circ \mathcal{T}^{s,\tau}$.*

Ainsi, la famille $\widetilde{\mathcal{T}}^\tau(s,u) = (s+\tau, \mathcal{T}^{s,s+\tau}u)$ est un groupe à un paramètre agissant sur l'ensemble $\bigcup_{\tau\in\mathbb{T}^1}\{\tau\}\times\Omega_\tau(\mathcal{T}) \subset \mathbb{T}^1 \times \mathcal{C}(M,\mathbb{R})$. Ceci permet d'associer à tout élément $u$ de $\Omega(\mathcal{T})$, une solution de viscosité $u(t,x) = \mathcal{T}^t u(x) : \mathbb{R}\times M \to \mathbb{R}$ définie en tous temps $t \in \mathbb{R}$, aussi dite solution de viscosité *globale*. De manière équivalente, si $u \in \Omega(\mathcal{T})$, il est récurrent par la proposition 4.1.1 et il existe suite strictement croissante d'entiers $p_n \in \mathbb{N}$ telle que $\mathcal{T}^{p_n}u$ converge vers $u$. Nous montrons que pour tout temps $t \in \mathbb{R}$, la solution de viscosité globale $u(t,x)$ associée à $u$ s'écrit aussi

$$u(t,x) = \mathcal{T}^t u(x) = \liminf_n \mathcal{T}^{p_n+t}u(x) \tag{4.1.3}$$

En particulier, le choix de valeur critique de Mañé et la proposition 2.1.1 montrent que la famille $\mathcal{T}^{p_n+t} = \mathcal{T}^t_{\alpha_0}u$ est uniformément bornée, et donc que toute solution de viscosité globale $u(t,x)$ de condition initiale non-errante $u(0,\cdot) \in \Omega(\mathcal{T})$ est bornée.

On note $\mathcal{B}(\mathcal{T}) \subset \mathcal{C}(M,\mathbb{R})$ l'ensemble des conditions initiales des solutions de viscosité globales bornées. Nous venons de voir l'inclusion $\Omega(\mathcal{T}) \subset \mathcal{B}(\mathcal{T})$. Et en travaillant sur l'$\alpha$-limite $\alpha(u)$ d'un élément $u$ de $\mathcal{B}(\mathcal{T})$, nous déduisons de la faible contractivité de $\mathcal{T}$ que $\alpha(u) = \omega(u)$ et que $u \in \omega(u)$, ce qui donne l'inclusion inverse $\mathcal{B}(\mathcal{T}) \subset \Omega(\mathcal{T})$. Ainsi, on obtient une nouvelle caractérisation des solutions de viscosité non-errantes en tant que solutions de viscosité globales bornées.

**Théorème 4.1.1.** *Une solutions de viscosité globale de l'équation de Hamilton-Jacobi (3.0.2) est bornée si et seulement si elle est non-errante. En d'autres termes, on a l'égalité d'ensembles suivante*

$$\Omega(\mathcal{T}) = \mathcal{B}(\mathcal{T}) \tag{4.1.4}$$

## 4.2 Formules de représentation de $\Omega(\mathcal{T})$

Dans la continuité de ce qui précède, nous nous proposons de décrire les éléments de l'ensemble non-errant $\Omega(\mathcal{T})$ via une formule de représentation. Il a été établi par

G.Contreras, R.Iturriaga et H.Sánchez-Morgado [CISM13] que toute solution KAM-faible $u$ de Fix$(\mathcal{T})$ s'écrit

$$u(x) = \inf_{y \in \mathbb{M}} \{\psi(y) + h^\infty(y,x)\} \tag{4.2.1}$$

où $\psi : \mathbb{M} \to \mathbb{R}$ est l'application $\psi = u_{|\mathbb{M}}$ et $\mathbb{M}$ est un sous-ensemble dans l'ensemble de Mather $\mathcal{M}_0$ de représentants des classes statiques définies comme les classes de la relation d'équivalence

$$x \sim_1 y \iff d_1(x,y) := h^\infty(x,y) + h^\infty(y,x) = 0 \tag{4.2.2}$$

Voir la section 2.2 pour plus de détails.

On voit à travers cette formule que les barrière de Peierls $h^\infty$ constituent en quelques sortes le squelette des solutions KAM-faibles. De surcroît, elles vérifient

1. Pour tout $y$ dans $\mathcal{M}_0$, $h^\infty(y,\cdot)$ est une solution KAM-faible.
2. Pour tout dans $x$ dans $\mathcal{M}_0$, $h^\infty(x,x) = 0$.
3. *(Inégalité triangulaire)* Pour tous $x$, $y$ dans $\mathcal{M}_0$ et $z$ dans $M$

$$h^\infty(x,z) \le h^\infty(x,y) + h^\infty(y,z) \tag{4.2.3}$$

Les deux derniers points qui justifient que $\sim_1$ est une relation d'équivalence.

Une généralisation nécessite donc de définir une barrière de Peierls généralisée $\underline{k} : \mathcal{M}_0 \times M \to \mathbb{R}$ qui vérifie ces trois propriétés, en remplaçant *solution KAM-faible* par *solution non-errante* dans la première, et qui jouerait le rôle de squelette de tout élément non-errant de $\Omega(\mathcal{T})$.

### 4.2.1 Barrière de Peierls généralisée

Nous divisons la construction de la barrière généralisée en deux étapes. Nous commençons dans un premier temps par construire une barrière non-canonique $\underline{h} : \mathcal{M}_0 \times M \to \mathbb{R}$ définie sur un sous-ensemble dense de l'ensemble de Mather $\mathcal{M}_0$.

L'idée est de considérer le sous-ensemble $\tilde{\mathcal{M}}_0^R$ des points de $\tilde{\mathcal{M}}_0$ qui sont récurrents en temps (entiers) négatifs par le flot Lagrangien $\phi_L^{-1}$. Son projeté $\mathcal{M}_0^R$ sur $M$ est un sous-ensemble de $\mathcal{M}_0$. La théorie de l'ergodicité affirme que

**Proposition 4.2.1** ([Mat91])**.** *L'ensemble $\mathcal{M}_0^R$ est dense dans $\mathcal{M}_0$.*

Ainsi, on définit les premières barrières de Peierls

**Définition 4.2.1.** 1. Pour toute suite strictement croissante $\underline{p} = (p_n)_{n\geq 0}$ d'entiers de $\mathbb{N}$, on définit la *$\underline{p}$-barrière de Peierls* $h^{\underline{p}} : M \times M \to \mathbb{R}$ par

$$h^{\underline{p}}(x,y) = \liminf_n h^{p_n}(x,y) \tag{4.2.4}$$

avec la dépendance temporelle

$$h^{\underline{p}+t}(x,y) = h^{\underline{p}}(t,x,y) = \liminf_n h^{t+p_n}(x,y) \tag{4.2.5}$$

où $h^t$ est le potentiel défini dans (2.2.1).

2. Pour tout point $x$ de l'ensemble de Mather récurrent $\mathcal{M}_0^R$ de relevé $\tilde{x}$ dans $\tilde{\mathcal{M}}_0^R$, on fixe une suite strictement croissante $\underline{p}^x = (p_n^x)_{n\geq 0}$ d'entiers de $\mathbb{N}$ telle $\tilde{x}$ est $(-\underline{p}^x)$-récurrente par le flot Lagrangien $\phi_L$.

   On définit la barrière $\underline{h} : \mathcal{M}_0^R \times M \to \mathbb{R}$ par

$$\underline{h}(x,y) = h^{\underline{p}^x}(x,y) = \liminf_n h^{p_n^x}(x,y) \tag{4.2.6}$$

   avec la dépendance temporelle

$$\underline{h}^t(x,y) = \underline{h}(t,x,y) = h^{\underline{p}^x+t}(x,y) \tag{4.2.7}$$

Nous montrons que cette barrière $\underline{h}$ vérifie deux des propriétés souhaitées, qui sont

1. Pour tout $y$ dans $\mathcal{M}_0^R$, $\underline{h}(\cdot,y,\cdot)$ est une solution de viscosité non-errante, i.e $\underline{h}(y,\cdot) \in \Omega(\mathcal{T})$.
2. Pour tout $x$ dans $\mathcal{M}_0^R$, $\underline{h}(x,x) = 0$.

Une autre propriété supplémentaire qui se révèlera importante dans la suite est que si $x_n$ est une suite de points de $M$ qui converge vers $x \in M$, alors

$$\lim_n \underline{h}(x_n,x) = \lim_n \underline{h}(x,x_n) = 0 \tag{4.2.8}$$

Noter que ceci reste vrai même si $x_n$ est prise en première variable, dont la dépendance de $\underline{h}$ n'est pas continue à cause des choix des suites $p^{x_n}$.

Un problème est que la barrière $\underline{h}$ n'est pas canonique et dépend fortement de suites $\underline{p}^x$ choisies dans la définition. De plus, la discontinuité en la première variable constitue un obstacle pour une extension naturelle à l'ensemble de Mather $\mathcal{M}_0$ tout entier. Et en supplément, $\underline{h}$ ne vérifie pas l'inégalité triangulaire (4.2.11).

Afin de forcer cette inégalité triangulaire, on définit la barrière de Peierls généralisée comme suit

**Définition 4.2.2.** On définit la *barrière de Peierls généralisée* $\underline{k} : \mathcal{M}_0^R \times M \to \mathbb{R}$ par

$$\underline{k}(x,y) = \inf\left\{\sum_{i=0}^{N-1} \underline{h}(x_i, x_{i+1}) \mid x_0 = x,\ x_N = y,\ x_i \in \mathcal{M}_0^R,\ N \geq 1\right\} \tag{4.2.9}$$

avec la dépendance temporelle

$$\underline{k}(t,x,y) = \underline{k}^t(x,y) = \inf\left\{\sum_{i=0}^{N-2} \underline{h}(x_i, x_{i+1}) + \underline{h}^t(x_{N-1}, y) \mid x_0 = x,\ x_N = y,\ x_i \in \mathcal{M}_0^R,\ N \geq 1\right\} \tag{4.2.10}$$

Ainsi, la nouvelle barrière $\underline{k}$ vérifie l'inégalité triangulaire, et pour tous $x$, $y$ dans $\mathcal{M}_0^R$ et $z$ dans $M$

$$\underline{k}(x,z) \leq \underline{k}(x,y) + \underline{k}(y,z) \tag{4.2.11}$$

En particulier, si $x_n$ et $y_n$ sont deux suites de points qui convergent vers $x$ et $y$, on obtient

$$\begin{cases} \underline{k}(x,y) - \underline{k}(x_n,y_n) \leq \underline{k}(x,x_n) + \underline{k}(y_n,y) \leq \underline{h}(x,x_n) + \underline{h}(y_n,y) \longrightarrow 0 & \text{quand } n \to +\infty \\ \underline{k}(x_n,y_n) - \underline{k}(x,y) \leq \underline{k}(x_n,x) + \underline{k}(y,y_n) \leq \underline{h}(x_n,x) + \underline{h}(y,y_n) \longrightarrow 0 & \text{quand } n \to +\infty \end{cases}$$

où les limites résultent de l'identité (4.2.8). On en déduit l'égalité $\lim_n \underline{k}(x_n,y_n) = \underline{k}(x,y)$ et par conséquent la continuité de la barrière généralisée $\underline{k}$.

Plus généralement, nous montrons que cette barrière est uniformément continue sur $\mathcal{M}_0^R \times M$ et nous en déduisant par densité de $\mathcal{M}_0^R$ dans l'ensemble de Mather $\mathcal{M}_0$ la proposition suivante.

**Proposition 4.2.2.** *La barrière de Peierls généralisée $\underline{k}$ s'étend de manière unique sur $\mathcal{M}_0 \times M$.*

*Et pour tout $y \in \mathcal{M}_0$, l'application $\underline{k}(\cdot, y, \cdot)$ est une solution de viscosité non-errante de l'équation de Hamilton-Jacobi (3.0.2), et $\underline{k}(y,\cdot)$ appartient à $\Omega(\mathcal{T})$.*

Il n'est pas clair que cette barrière est indépendante du choix des suites $\underline{p}^x$ utilisées pour définir la barrière non canonique $\underline{h}$. Ceci sera une conséquence de la formule de représentation de $\Omega(\mathcal{T})$.

### 4.2.2 Formules de Représentations et conséquences

Maintenant que nous avons construit la barrière de Peierls généralisée $\underline{k}$, nous pouvons définir les classes statiques généralisées associée par

**Définition 4.2.3.** L'ensemble $\underline{\mathbb{M}}$ des *classes statiques généralisées* est l'ensemble des classes de la relation d'équivalence $\sim$ sur $\mathcal{M}_0$ définie par

$$x \sim y \quad \Longleftrightarrow \underline{d}(x,y) = \underline{k}(x,y) + \underline{k}(y,x) = 0 \tag{4.2.12}$$

Nous introduisons aussi la notion de domination suivante

**Définition 4.2.4.** Soit $X$ un ensemble et soit $f : X \times X \to \mathbb{R}$ une application. Une application $\psi : X \to \mathbb{R}$ est dite *$f$-dominée sur $X$* si pour tous $x$ et $y$ dans $X$, on a

$$\psi(y) - \psi(x) \leq f(x,y) \tag{4.2.13}$$

On note $\mathrm{Dom}(X,f)$ l'ensemble des applications $f$-dominées sur $X$.

Le résultat principal du chapitre est le suivant

**Théorème 4.2.5.** *L'application $\Psi_{\underline{k}}$ suivante est une bijection*

$$\begin{array}{rcl} \Psi_{\underline{k}} : \mathrm{Dom}(\mathbb{M}, \underline{k}) & \longrightarrow & \Omega(\mathcal{T}) \\ \psi & \longmapsto & \inf\limits_{y \in \mathbb{M}} \{\psi(y) + \underline{k}(y, \cdot)\} \end{array} \tag{4.2.14}$$

*d'inverse l'application restriction*

$$\begin{array}{rcl} \Phi_{\underline{k}} : \Omega(\mathcal{T}) & \longrightarrow & \mathrm{Dom}(\mathbb{M}, \underline{k}) \\ v & \longmapsto & v_{|\mathbb{M}} \end{array} \tag{4.2.15}$$

Cette nouvelle barrière $\underline{k}$ est bien le squelette de toutes les solutions de viscosité non-errantes dans le cas non-autonome. Et puisque nous avons déjà remarqué que $\underline{k}(y,\cdot)$ appartient à $\Omega(\mathcal{T})$, nous pouvons identifier cet élément comme le suprémum des solutions de viscosité non-errantes qui s'annule en $y$.

**Corollaire 4.2.6.** *Pour tous $x_0 \in \mathcal{M}_0$ et $x \in M$, nous avons la formule*

$$\underline{k}(x_0, x) = \max_{v \in \Omega(\mathcal{T})} \{v(x) - v(x_0)\} \tag{4.2.16}$$

Il en résulte que la barrière $\underline{k}$ est canonique et ne dépend pas des suites $\underline{p}^x$ nécessaires à la définition de $\underline{h}$.

Cette formule de représentation établit aussi lien entre la structure des solutions non-errantes et la dynamique du flot Lagrangien restreint à l'ensemble de Mather. Quelques implications de ceci sont résumées dans les deux corollaires ci-dessous.

Dans le cas où l'ensemble de Mather $\mathcal{M}_0$ est constitué uniquement de courbes périodiques de périodes (non nécessairement minimales) $N \geq 1$, alors on peut montrer que la barrière de Peierls généralisée $\underline{k}$ et que la $N$-barrière $h^{N\infty}$ (Définie dans (3.2.1)) coïncident, et $\underline{k} = h^{N\infty}$ sur $\mathcal{M}_0 \times M$. Il en résulte que

**Corollaire 4.2.7.** *S'il existe un entier positif $N \geq 1$ tel que $\phi^N_{L|\mathcal{M}_0} = Id_{\mathcal{M}_0}$, alors $\Omega(\mathcal{T}) = \mathrm{Per}_N(\mathcal{T})$.*

De manière similaire, s'il existe une suite strictement croissante d'entier $\underline{p} = (p_n)_n$ telle que tous les points de l'ensemble de Mather sont uniformément récurrents pour cette suite et par le flot Lagrangien $\phi_L$, alors $\underline{k} = h^{\underline{p}}$ et il en résulte que

**Corollaire 4.2.8.** *S'il existe une suite strictement croissante d'entiers positifs $\underline{p} = (p_n)_{n\geq 0}$ telle que $\phi_{L|\tilde{\mathcal{M}}_0}^{-p_n}$ converge uniformément vers l'identité, alors les éléments $v$ de $\Omega(\mathcal{T})$ sont $\underline{p}$-récurrents i.e $\lim_n \mathcal{T}^{p_n} v = v$ uniformément sur $v$.*

## 4.3 Construction d'une solution de viscosité récurrente non-périodique

Nous avons étudié l'ensemble $\Omega(\mathcal{T})$ et donné une description de ses éléments. Cependant, rien ne prouve pour l'instant qu'il existe des Hamiltoniens pour lesquels $\Omega(\mathcal{T}) \neq \mathrm{Per}(\mathcal{T})$. En effet, P. Bernard et J.-M. Roquejoffre [BR04] ont montré que ces deux ensembles sont égaux si la variété $M$ est de dimension 1.

Dans le chapitre 8, nous répondons à cette question et nous construisons sur toute variété compacte connexe $M$ de dimension $d \geq 2$, un Hamiltonien de Tonelli possédant des solutions de viscosité récurrentes et non périodiques en temps. De plus, nous renforçons ce résultat en obtenant une solution de régularité $C^\infty$.

**Théorème 4.3.1.** *Pour toute variété $M$ de dimension $d \geq 2$, il existe un Hamiltonien de Tonelli $C^\infty$ régulier $H : \mathbb{T}^1 \times T^*M \to \mathbb{R}$ dont le semi-groupe de Lax-Oleinik $\mathcal{T}$ admet un élément $C^\infty$-régulier, récurrent et non-périodique.*

L'idée de la construction (non régulière) est de considérer une suite strictement croissante d'entiers positifs $\rho_n \geq 2$, et une suite de solutions de viscosité $\rho_n$-périodiques données par les $\rho_n$-barrières $h^{\rho_n\infty}(x_n, \cdot)$ définies dans (3.2.1). Ensuite, nous regardons la solution de viscosité $u : M \to \mathbb{R}$ donnée par

$$u(x) = \inf_{n\geq 0}\{h^{\rho_n\infty}(x_n, x)\} \tag{4.3.1}$$

Cette forme est inspirée de la formule de représentation (4.2.14), qui produit des solutions de viscosité non-errantes de $\Omega(\mathcal{T})$. On peut aussi constater que $u \in \Omega(\mathcal{T})$ en remarquant qu'elle est globalement définie et bornée, puis en utilisant la caractérisation des solutions de viscosité non-errantes prouvée dans le théorème 4.1.1.

Afin d'éviter que $u$ ne soit périodique, il faut faire en sorte que les barrières $h^{\rho_n\infty}(x_n, \cdot)$ soient de périodes minimales $\rho_n$, et que pour tout $n$, il y ait une région de la variété $M$ où l'infimum dans (4.3.1) soit réalisé par cette barrière $h^{\rho_n\infty}(x_n, \cdot)$. Cela permet d'obtenir des points $x_n'$ d'orbites $(u(t, x_n'))_{t\geq 0}$ périodiques de périodes minimales $\rho_n$. Et puisque ces périodes $\rho_n$ divergent vers l'infini, la périodicité de $u$ devient impossible.

Une telle construction nécessite de pouvoir comparer les différentes barrières $h^{\rho_n\infty}(x_n, x)$ sur la variété $M$. C'est ce que nous faisons en considérant une famille de Lagrangiens qui permettent de simplifier cette étude.

### 4.3.1 Lagrangiens de Mañé

Une famille de Lagrangiens $L : \mathbb{T}^1 \times TM \to \mathbb{R}$ qui permet une étude simple de son flot Lagrangien et de ses barrières de Peierls est la famille des Lagrangiens de Mañé introduits par R. Mañé dans l'appendice de [Mn92].

**Définition 4.3.2.** Soit $f_t$ une isotopie de $M$ de champs de vecteurs $X_t$ 1-périodique en temps. Le *Lagrangien de Mañé* $L : \mathbb{T}^1 \times TM \to \mathbb{R}$ de Tonelli associé à $f_t$ est défini par

$$L(t, x, v) = \frac{1}{2}\|v - X_t(x)\|^2 \tag{4.3.2}$$

Le carré de la norme fournit la convexité et la surlinéarité demandées dans la définition de Tonelli 2.0.1. Ainsi, il est possible d'associer à $L$, par conjugaison convexe, un Hamiltonien de Mañé $H : \mathbb{T}^1 \times T^*M \to \mathbb{R}$

$$H(t, x, p) = \frac{1}{2}\|p + X_t(x)\|^2 - \frac{1}{2}\|X_t(x)\|^2 \tag{4.3.3}$$

Le sous-ensemble $\{v = X_t(x)\}$ du fibré tangent $TM$ est la région où le Lagrangien $L$ est nul et donc minimal. C'est donc cet ensemble qui contient les informations intéressantes pour la théorie KAM-faible ; entre autres, il inclut les ensembles de Mather, d'Aubry, ainsi que les courbes statiques calibrées par toutes les solutions de viscosité non-errante.

Rappelons que la transformation de Legendre $\mathcal{L} : \mathbb{T}^1 \times TM \to \mathbb{T}^1 \times T^*M$ définie par $\mathcal{L}(t, x, v) = (t, x, \partial_v L(t, x, v))$, conjugue le flot Lagrangien $\phi_L$ et le flot Hamiltonien $\phi_H$. Nous obtenons

**Proposition 4.3.1.**
1. *La transformée de Legendre $\mathcal{L}$ envoie l'ensemble $\{v = X_t(x)\}$ sur la section nulle $0_{T^*M}$ de $T^*M$.*
2. *La restriction du flot Hamiltonien $\phi_H$ à la section nulle de $T^*M$ est égale à l'isotopie $f_t$.*

Cette proposition montre que le tracé de $f_t$, choisi à notre convenance, détermine le flot Hamiltonien $\varphi_H$ restreint à la section nulle de $T^*M$, qui est elle-même envoyée par la transformée de Legendre vers l'ensemble $v = X_t(x)$, où toute l'information importante à propos des barrières de Peierls réside. Le bon choix du flot $f_t$ est l'élément clé de cette construction. Cependant, nous devons expliquer le comportement de $h^\infty$ avant de présenter notre choix.

Le Lagrangien $L$ est positif. Ce fait simplifie fortement la compréhension du comportement des barrières de Peierls $h^{\rho_n\infty}$ et permet de les interpréter en utilisant la notion de pseudo-orbites.

**Définition 4.3.3.** Pour tous $\varepsilon > 0$ et $\tau > 0$ fixés, et pour tous points $x$ et $y$ de $M$, une $(\varepsilon, \tau)$-*pseudo-orbite* du flot de $X_t$ entre $x$ et $y$ est une famille de courbes $(\gamma_k : [S_k, T_k] \to M)_{0\le k\le m}$ telle que

i. $S_0 = 0$ et $\gamma_0(0) = x$.
ii. Pour tout $0 \le k \le m$, $\dot\gamma_k(t) = X_t(\gamma_k(t))$.
iii. Les temps réels $S_k$ et $T_k$ verifient $T_k - S_k \ge \tau$ et $S_{k+1} = T_k \; mod \, 1$.
iv. Pour tout $0 \le k \le m-1$, $d\big(\gamma_k(T_k), \gamma_{k+1}(S_{k+1})\big) < \varepsilon$ et $d\big(\gamma_m(T_m), y\big) < \varepsilon$.

Ainsi, nous montrons le résultat suivant qui détecte quand ces barrières de Peierls sont strictement positives.

**Proposition 4.3.2.** *Si pour $\tau > 0$ fixé, il existe un $\varepsilon > 0$ tel qu'il n'y ait pas de $(\varepsilon, \tau)$-pseudo-orbite de $X_t$ entre $x$ et $y$, alors* $\liminf_{t\to+\infty} h^t(x, y) > 0$.

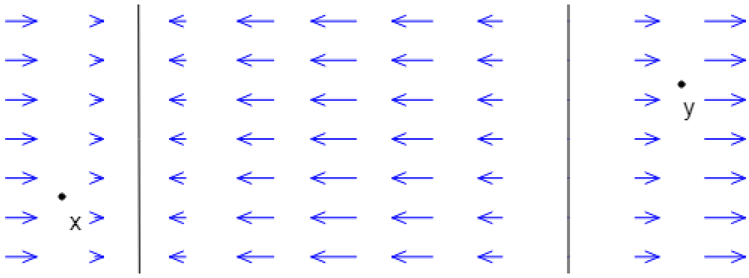
En d'autres termes, on peut détecter l'annulation des barrières de Peierls à partir de la dynamique de $f_t$ comme indiqué dans la figure 4.1 ci-dessous. Noter que pour évaluer la $\rho_n$-barrière $h^{\rho_n\infty}(x, y)$, il faut prendre des pseudo-orbites avec des temps $S_k$ et $T_k$ qui sont multiples de la période $\rho_n$. Ceci introduit une subtilité supplémentaire dans la compréhension de $h^{n\infty}$ pour laquelle il faut prendre en considération l'évolution temporelle de $f_t$. Ceci est représenté dans la figure 4.1c.

En particulier, tout point récurrent par chaînes (avec un nombre fini d'orbites) est un point de l'ensemble d'Aubry $\mathcal{A}_0$.

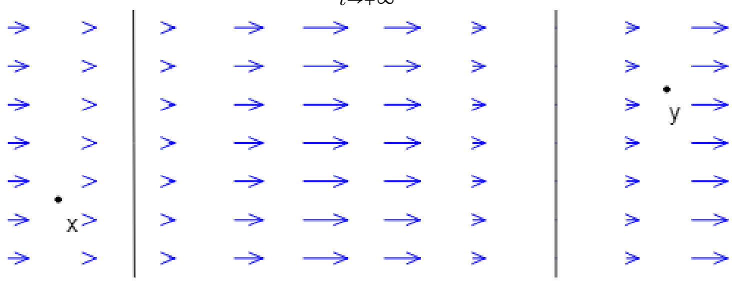
### 4.3.2 Construction d'une solution de viscosité récurrente, non-périodique

On se place sur une carte de $B \subset \mathbb{R}^d$ de $M$, de coordonnées $(x_1, .., x_d)$ et on note $(r, \theta)$ les coordonnées polaires du plan $(x_1, x_2)$. On considère deux suites strictement décroissantes de rayons $r_n$ et $\delta_n \ll r_n$ qui convergent vers 0 et on définit les ensembles

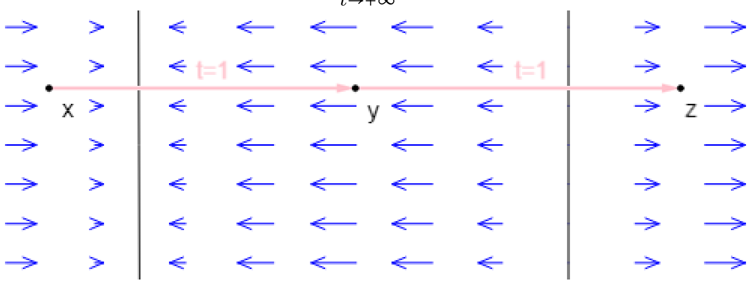
$$\begin{aligned} O_n &= \{x = (r, \theta, x_3, .., x_d) \; ; \; r = r_n\} \\ C_n &= \{x \in B \; ; \; d(x, O_n) < 2\delta_n\} \end{aligned} \tag{4.3.4}$$




(a) Cas où $\liminf\limits_{t\to+\infty} h^t(x,y) > 0$.




(b) Cas où $\liminf\limits_{t\to+\infty} h^t(x,y) = 0$.




(c) Prise en compte de la translation temporelle. Nous avons $h^\infty(x,y) = h^\infty(x,z) = 0$, mais $h^{2\infty}(x,y) > 0$ et $h^{2\infty}(x,z) = 0$.

FIGURE 4.1 – Évaluation graphique de $\liminf\limits_{t\to+\infty} h^t(x,y)$ et $h^{n\infty}(x,y)$.

On choisit la dynamique du champs de $X_t$ dans les ensembles $C_n$ comme étant la composition d'un champs de vecteurs autonome $Z$ représenté dans la figure 8.1, composé avec une rotation $\mathcal{R}$ d'angle $1/\rho_n$. Ainsi, l'orbite $\{x_n^i = f_i(x_n)\}$ du point $x_n := (r_n, 0, .., 0)$ est $\rho_n$-périodique et répulsive pour le champ $X_t$.

Une solution récurrente non-périodique est alors donnée par la solution de viscosité $u : \mathbb{R} \times M \to \mathbb{R}$ de condition initiale

$$u(x) = \inf_{n \geq 0} \{h^{\rho_n \infty}(x_n, x)\} \tag{4.3.5}$$

**Récurrence.** La récurrence, ou la non-errance, de la solution de viscosité $u$ est une conséquence du théorème 4.1.1 qui caractérise les éléments de $\Omega(\mathcal{T})$ comme étant l'ensemble des solutions de viscosité globales et bornées.

Dans le chapitre, nous montrons cette récurrence explicitement en montrant que $\lim_n \mathcal{T}^{p_n} u = u$ pour la suite d'entiers $p_n = \prod_{k=0}^n \rho_k$. Nous arrivons même à décrire précisément son ensemble $\omega$-limite $\omega(u)$.

**Non-périodicité.** Pour la non-périodicité, il nous suffit de voir que les solutions de viscosité $\rho_n$-périodique $h^{\rho_n \infty}(x_n, \cdot)$ vérifient

(i) Pour $k = 1, ..., \rho_n - 1$, $\mathcal{T}^k h^{\rho_n \infty}(x_n, \cdot)(x_n) = h^{\rho_n \infty + k}(x_n, x_n) > 0$.

(ii) Pour $k = 0$ ou $\rho_n$, $\mathcal{T}^k h^{\rho_n \infty}(x_n, \cdot)(x_n) = h^{\rho_n \infty}(x_n, x_n) = 0$.

En effet, l'orbite $f_t(x_n)$ de $x_n$ est $\rho_n$-périodique et d'action nulle par le Lagrangien $L$ car

$$L(t, f_t(x_n), \dot{f}_t(x_n)) = \frac{1}{2} \|\dot{f}_t(x_n) - X_t(f_t(x_n))\|^2 = 0$$

Ceci qui donne $h^{\rho_n i}(x_n, x_n) = 0$ pour tout entier $i \geq 1$ et

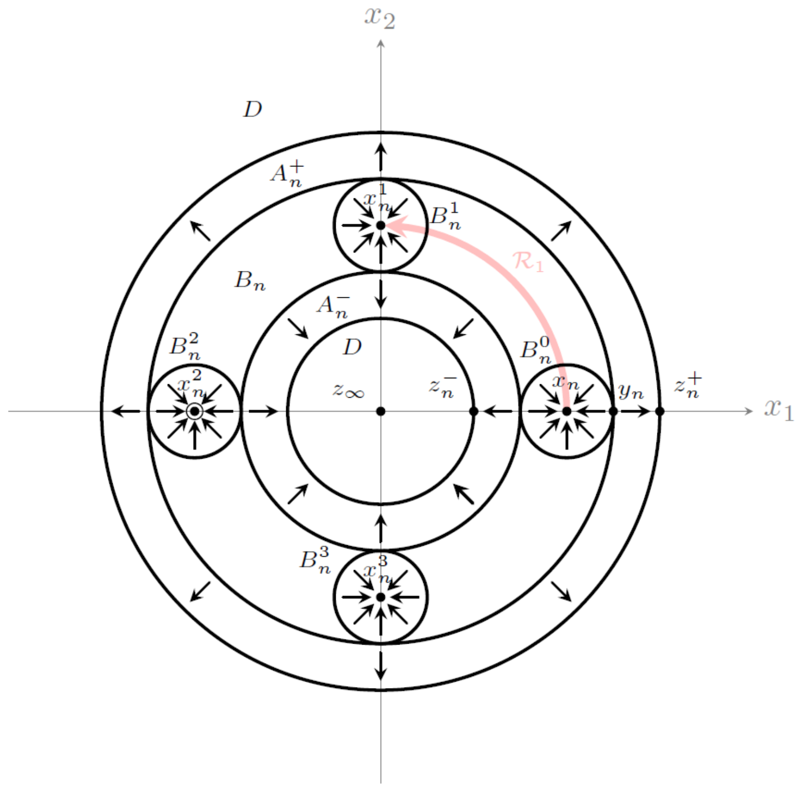
$$h^{\rho_n \infty}(x_n, x_n) = \liminf_i h^{\rho_n i}(x_n, x_n) = 0$$

D'autre part, puisque l'orbite $f_t(x_n)$ de $x_n$ est répulsive, nous pouvons montrer à travers la proposition 4.3.2 que $h^{\rho_n \infty + k}(x_n, x_n) > 0$ dès que $k$ n'est pas un multiple de la période $\rho_n$.

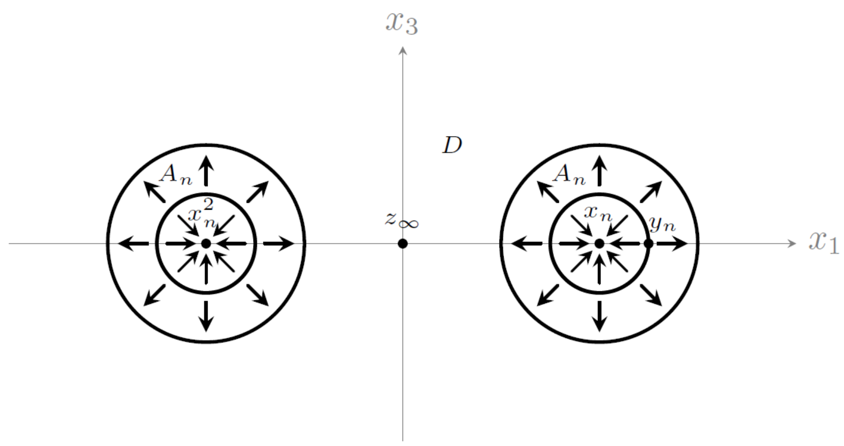
Finalement, nous montrons la formule

$$u(k, x) = \mathcal{T}^k u(x) = \inf_{n \geq 0} \{\mathcal{T}^k h^{\rho_n}(x_n, x)\} = \inf_{n \geq 0} \{h^{\rho_n + k}(x_n, x)\} \tag{4.3.6}$$

qui vérifie pareillement




(a) Vue du dessus de $C_n$.




(b) Vue de profil de $C_n$.

FIGURE 4.2 – Champs de vecteur $X_t$ sur $C_n$.

(i) Pour $k = 1, ..., \rho_n - 1$, $\mathcal{T}^k u(x_n) > 0$.

(ii) Pour $k = 0$ ou $\rho_n$, $\mathcal{T}^k u(x_n) = 0$.

Noter que le premier point nécessite plus de travail et ne se déduit pas directement de ce qui est dit au-dessus pour $h^{\rho_n + k}(x_n, \cdot)$.

Puisque la suite des périodes $\rho_n$ diverge vers l'infini, la périodicité de $u$ est impossible.

### 4.3.3 Construction d'une solution $C^\infty$-régulière

L'obtention d'une solution régulièrement est hautement plus difficile et nécessite d'avoir des symétries précises dans la construction du champs de vecteurs $X_t$ et qui n'étaient pas essentielles pour assurer $\Omega(\mathcal{T}) \setminus \mathrm{Per}(\mathcal{T}) \neq \emptyset$. Le champs $X_t$ représenté dans la figure 8.1 permet malgré tout d'avoir $\big(\Omega(\mathcal{T}) \setminus \mathrm{Per}(\mathcal{T})\big) \cap \mathcal{C}^\infty(M, \mathbb{R}) \neq \emptyset$. Pour obtenir une condition initiale régulière, nous modifions le choix de la condition initiale $u$ définie dans (4.3.5).

Une application de la formule de représentation (4.2.14) de l'ensemble non-errant $\Omega(\mathcal{T})$ montre que les conditions initiales de la forme

$$u_c(x) = \inf_{n \geq 0} \{c_n + h^{\rho_n \infty}(x_n, x)\}$$

pour des constantes $c_n \in \mathbb{R}$, sont toujours des solutions de viscosité récurrentes. Nous nous proposons alors de montrer qu'il existe un bon choix de constantes $c_n$ qui rendent la solution $u_c$ de $C^\infty$-régulière.

Pour ce faire, il faut réussir à éviter les irrégularités causées par l'infimum. Une idée pour contourner cette difficulté est de profiter de la différentiabilité des solutions de viscosité sur les courbes calibrées mentionnée dans la proposition 2.1.2. En effet, nous démontrons dans le premier travail sur les formules des représentations (chapitre 7) la proposition suivante

**Proposition 4.3.3.** *Soit $x$ un point de l'ensemble de Mather et soit $\gamma : \mathbb{R} \to M$ la courbe de l'ensemble de Mather définie $\gamma(t) = \pi \circ \phi_L(\tilde{x})$ où $\tilde{x}$ est le relevé de $x$ à $\tilde{\mathcal{M}}$. Alors la courbe $\gamma$ est calibrée par toute solution de viscosité récurrente $u$ de $\Omega(\mathcal{T})$.*

Ainsi, on déduit que la solution de viscosité $u_c$ est toujours différentiable sur l'ensemble de Mather $\mathcal{M}_0$. Nous choisissons alors les constantes $c_n$ de manière à ce que les frontières entre les infimums soient réalisées sur l'ensemble de Mather $\mathcal{M}_0$.

Ceci réduit l'étude de la régularité à celle des barrières de Peierls $h^{\rho_n \infty}(x_n, x)$ sur les différentes régions de $M \setminus \mathcal{M}_0$. La difficulté est que ces barrières ne sont en général pas régulières et leurs différentielles peuvent présenter des discontinuités comme nous pouvons le constater pour la solution KAM-faible du pendule simple, représenté dans la figure 2.2b, la raison étant qu'il existe un point $x_{1/2}$ (point de discontinuité) possédant deux courbes calibrées différentes $\gamma_1$ et $\gamma_2$ telles que $\gamma_1(0) = \gamma_2(0) = x_{1/2}$ et $\dot\gamma_1(0) \neq \dot\gamma_2(0)$, l'une portée par la partie supérieure, et l'autre par la partie inférieure du niveau critique $\{H = 1\}$. Ainsi, en suivant la formule de la différentielle (2.1.9) sur les courbes calibrées, les formes $\partial_v L(0, x_{1/2}, \dot\gamma_1(0)$ et $\partial_v L(0, x_{1/2}, \dot\gamma_2(0)$ seraient deux candidats possibles pour $d_{x_{1/2}} h^\infty(x_0, \cdot)$, ce qui impliquerait la non-différentiabilité de $h^\infty(x_0, \cdot)$ en $x_{1/2}$.

Nous contournons cette difficulté en imposant des symétries radiales au champ autonome $Z$ représenté dans la figure 8.1. Sous ces symétries, nous sommes capable d'identifier les courbes calibrées des barrières $h^{\rho_n \infty}(x_n, x)$ dans différentes régions. Nous rappelons que l'isotopie $f_t$ du champs $X_t$ est la composée d'une isotopie autonome $g_t$ de champ de vecteur $Z$ et d'une rotation $\mathcal{R}_t$ d'angle $t/\rho_n$ dans $C_n$. Nous démontrons alors la proposition suivante.

**Proposition 4.3.4.** *Dans les différentes composantes connexes de $M \smallsetminus \mathcal{M}_0$, nous avons deux cas de figure :*

1. *Soit $u_c$ est constante.*
2. *Soit pour tout point $x$ de cette région, la courbe $\mathcal{R}_t \circ \phi^t_{-Z}(x)$ est calibrée par $u_c$.*

Et nous en déduisons par la formule (2.1.9) que dans les régions où $u_c$ n'est pas constante, nous avons

$$du_c(x) = \partial_v L\left(0, x, \frac{d}{dt}\Big|_{t=0} \left(\mathcal{R}_t \circ \phi^t_{-Z}(x)\right)\right) \tag{4.3.7}$$

qui est $C^\infty$-régulière. Et si le champs de vecteur $Z$ et toutes ses dérivées s'annulent sur les bords de $\mathcal{M}_0$, ce sera aussi le cas pour $du_c$. Ainsi, la $C^\infty$-régularité de $u_c$ s'étend sur la variété $M$ tout entière.

## 4.4 Théorème de Birkhoff pour les Lagrangiennes récurrentes

Dans le chapitre 9, nous nous penchons sur un théorème de Birkhoff multidimensionnel dont la preuve fait intervenir des techniques venant de la théorie KAM-faible et des propriétés des solutions de viscosité non-errantes $\Omega(\mathcal{T})$.

Un théorème dû à Birkhoff (Voir section 1.2) affirme qu'une courbe essentielle invariante par un difféomorphisme du cylindre qui dévie la verticale est un graphe Lipschitz au-dessus du cercle. Une généralisation par M.-C. Arnaud et A. Venturelli [AV17] étend ce résultat aux Hamiltoniens de Tonelli sur une variété compacte connexe $M$. Ils montrent qu'une sous-variété Lagrangienne du fibré cotangent $T^*M$ d'une variété compacte connexe $M$, qui est H-isotope à la section nulle et invariante par le temps 1 d'un flot Hamiltonien $\phi_H$ de Tonelli, est un graphe Lipschitz au-dessus de la section nulle.

L'idée est de notre résultat est d'étendre ce théorème de Birkhoff au-delà des Lagrangiennes invariantes. Nous incluons premièrement les sous-variétés Lagrangiennes récurrentes, et secondement celles qui convergent en temps positifs et négatifs vers d'autres sous-variétés Lagrangiennes sans se dilater, dans un sens que nous précisons à la définition 4.4.1. Nous montrons alors que de telles sous-variétés Lagrangiennes sont les graphes de différentielles de solutions de viscosité non-errante, i.e. d'éléments de $\Omega(\mathcal{T})$.

### 4.4.1 Énoncés des résultats

On se place dans le fibré cotangent $T^*M$ d'une variété $M$ compacte connexe sans bords de dimension $d$, muni de la forme symplectique standard $\omega = -d\lambda = dq \wedge dp$ et de la forme de Liouville $\lambda(q,p) = p \circ d\pi(q,p) = p.dq$.

Afin de définir la bonne notion de convergence sur les sous-variétés Lagrangiennes qui nous intéressent, nous considérons d'abord la distance de Hausdorff $d_H$ sur les ensemble compacts définie par

$$d_H(\mathcal{L},\mathcal{L}') = \max\left\{ \sup_{x' \in \mathcal{L}'} d(x',\mathcal{L}) \ , \ \sup_{x \in \mathcal{L}} d(x,\mathcal{L}') \right\} \tag{4.4.1}$$

La topologie induite par cette distance sur les sous-variétés Lagrangiennes compactes est ce qui s'apparente à une topologie $C^0$. Par exemple, une suite de 1-formes fermées $\eta_n$ converge en topologie $C^0$ vers $\eta$ si et seulement si les graphes de $\eta_n$ dans $T^*M$, qui sont des sous-variétés Lagrangiennes, convergent pour la distance de Hausdorff.

Rappelons qu'une sous-variété Lagrangienne $\mathcal{L}$ qui est $H$-isotope à la section nulle est exacte et possède une primitive de Liouville $h$ qui vérifie la relation $\lambda_{|T\mathcal{L}} = dh$. Le contrôle de la "dilatation" des sous-variétés Lagrangiennes exactes se fera à travers de *l'oscillation* de ces primitives $h$ définie par

$$\mathrm{Osc}(h) = \max h - \min h \tag{4.4.2}$$

Finalement, nous introduisons le *groupe Hamiltonien* $\mathrm{Ham}(T^*M,\omega)$ qui est l'ensemble des applications Hamiltoniennes de $(T^*M,\omega)$, c'est-à-dire l'ensemble des temps 1 $\phi_H^1$ de flots Hamiltoniens sur $(T^*M,\omega)$.

La notion de convergence que nous utiliserons est la suivante

**Definition 4.4.1.** Soit $(\mathcal{L}_n)_{n\geq 0}$ et $\mathcal{L}$ des ous-variétés Lagrangiennes de $T^*M$ H-isotopes à la section nulle $0_{T^*M}$. On dit que la suite $(\mathcal{L}_n)_{n\geq 0}$ *convergence avec complexité réduite* vers $\mathcal{L}$ si

i. $\lim_n d_H(\mathcal{L}_n), \mathcal{L}) = 0$.
ii. Si $\varphi \in \mathrm{Ham}(T^*M, \omega)$ est telle que $\mathcal{L} = \varphi(0_{T^*M})$, et si $l_n$ est une primitive de Liouville sur la sous-variété Lagrangienne $\varphi^{-1}(\mathcal{L}_n)$, alors $\lim_n \mathrm{Osc}(l_n) = 0$.

**Proposition 4.4.2.** *La définition de la convergence à complexité réduite ne dépend pas du choix de l'application Hamiltonienne $\varphi$ telle que $\mathcal{L} = \varphi(0_{T^*M})$.*

L'idée principale de cette définition est de réussir à définir une notion de convergence des sous-variétés Lagrangiennes $\mathcal{L}_n$ vers une sous-variété Lagrangienne $\mathcal{L}$ avec un contrôle sur la différence entre leurs primitives de Liouville. Cela permet de limiter la dilatation et l'enroulement des sous-variétés lagrangiennes $\mathcal{L}_n$ autour de $\mathcal{L}$ comme indiqué dans la figure 4.3. Ce phénomène n'est pas exclu par la convergence en topologie Hausdorff.

Cependant, des primitives de Liouville $h_n$ et $h$ sur $\mathcal{L}_n$ et $\mathcal{L}$ ont des domaines de définition différents. Et pour les comparer, on propose de se ramener au cas où $h$ est constante, c'est-à-dire lorsque $\mathcal{L}$ est la section nulle du fibré cotangent. Ceci est réalisé par l'application de $\varphi^{-1}$.

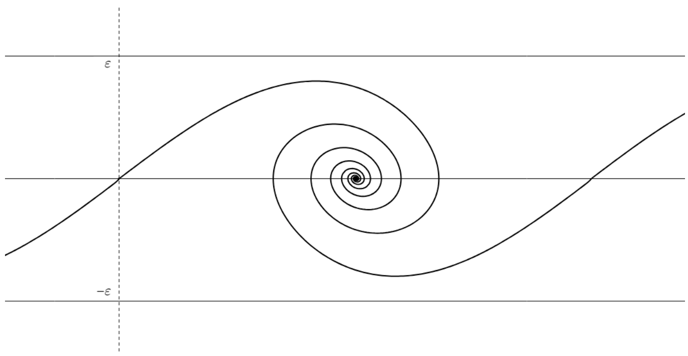



FIGURE 4.3 – Phénomène à éviter

Le résultat principal du chapitre est le...

**Theorem 4.4.3.** *Soit $H : \mathbb{T}^1 \times T^*M \to \mathbb{R}$ un Hamiltonien de Tonelli de flot $\phi_H$ et soit $\mathcal{L}$ une sous-variété Lagrangienne de $T^*M$ qui est $H$-isotope à la section nulle. Pour tout temps $t \in \mathbb{R}$, on pose $\mathcal{L}_t := \phi_H^t(\mathcal{L})$. S'il existe deux sous-variétés Lagrangiennes $\mathcal{L}_\omega$ et $\mathcal{L}_\alpha$ H-isotopes à la section nulle, et s'il existe deux suites strictement croissantes d'entiers positifs $n_k$ et $m_k$ telles que $(\mathcal{L}_{n_k})_{k\geq 0}$ et $(\mathcal{L}_{-m_k})_{k\geq 0}$ convergent respectivement, etavec complexités réduites, vers $\mathcal{L}_\omega$ et $\mathcal{L}_\alpha$, alors $\mathcal{L}$ et toutes ses images $\mathcal{L}_t$ sont des graphes $C^1$ au-dessus de la section nulle $0_{T^*M}$ de $T^*M$.*

Et par conséquent, le théorème de Birkhoff reste vrai pour les sous-variétés lagrangiennes $\mathcal{L}$ qui sont $\phi_H$-récurrentes avec convergence à complexité bornées vers elle-mêmes. Le corollaire s'énonce comme suit.

**Corollary 4.4.4.** *Soit $H : \mathbb{T}^1 \times T^*M \to \mathbb{R}$ un Hamiltonien de Tonelli de flot $\phi_H$ et soit $\mathcal{L}$ une sous-variété Lagrangienne de $T^*M$ qui est $H$-isotope à la section nulle. Si $\mathcal{L}$ est positivement et négativement récurrente avec convergence à complexité réduite, c'est-à-dire s'il existe deux suites strictement croissantes d'entiers positifs $n_k$ et $m_k$ telles que $(\mathcal{L}_{n_k})_k$ et $(\mathcal{L}_{-m_k})_k$ convergent avec complexités réduites vers $\mathcal{L}$, alors $\mathcal{L}$ et toutes ses images $\mathcal{L}_t$ sont des graphes $C^1$ au-dessus de la section nulle $0_{T^*M}$ de $T^*M$.*

Un autre corollaire serait une application aux sous-variétés Lagrangiennes périodiques ou invariantes, ce qui donne le théorème 2.3.1, ainsi que sa version non-autonome [AV17].

Par ailleurs, nous montrons aussi que le théorème 4.4.3 et son corollaire 4.4.4 sont équivalents dans le sens où : si deux sous-suites des sous-variétés Lagrangiennes $\phi_H^n(\mathcal{L})$ convergent en complexités réduites en temps positifs et négatifs alors nécessairement, la sous-variété $\mathcal{L}$ est récurrente pour cette même convergence, en temps négatifs et positifs, par le flot Hamiltonien $\phi_H$.

**Corollary 4.4.5.** *Sous les hypothèses du théorème 4.4.3, la sous-variété Lagrangienne $\mathcal{L}$ est $\phi_H^1$-récurrente pour la distance de Hausdorff.*

La raison de ceci résulte des propriétés des solutions $C^1$-régulières (et donc de viscosité) de l'équation de Hamilton-Jacobi (3.0.2). En effet, dans la preuve du théorème 4.4.3, nous trouvons une solution de viscosité $u(t,q)$ qui est $C^1$-régulière et telle que $\mathcal{L}_t = \phi_H^t(\mathcal{L})$ est le graphe de $d_q u(t,\cdot)$. Or, nous avons l'énoncé suivant qui se démontre par les outils de la théorie KAM-faible.

**Proposition 4.4.6.** *Pour un Hamiltonien de Tonelli $H : \mathbb{T}^1 \times T^*M \to \mathbb{R}$ sur une variété compacte connexe $M$, toute solution globale et $C^1$-régulière $u(t,q) : \mathbb{R} \times M \to \mathbb{R}$ de l'équation de Hamilton-Jacobi*

$$\partial_t u + H(t,q,d_q u) = \alpha_0$$

*est récurrente en temps négatifs et positifs. En particulier, nous avons $u(0,\cdot) \in \Omega(\mathcal{T})$.*

Nous obtenons donc que $\mathcal{L}$ est le graphe de la différentielle $du$ d'une solution de viscosité récurrente en temps négatifs et positifs. De plus, nous montrons que la différentielle $du$ est elle-même récurrente en temps positifs et négatifs avec une convergence à complexité réduite.

Dans le cas autonome, nous savons déjà, d'après le Théorème 2.1.2 de convergence de Fathi, que $\Omega(\mathcal{T}) = \mathrm{Fix}(\mathcal{T}) = \bigcap_{t\in\mathbb{R}} \mathrm{Fix}(\mathcal{T}^t)$. Ceci implique que toutes les solutions récurrentes de l'équation de Hamilton-Jacobi sont des solutions KAM faibles stationnaires indépendantes du temps. Il en résulte le corollaire suivant, qui est constitue une version autonome du corollaire 4.4.4.

**Corollary 4.4.7.** *Sous les hypothèses du théorème 4.4.3 et si $H : T^*M \to \mathbb{R}$ est autonome, alors la sous-variété Lagrangienne $\mathcal{L}$ est un graphe $C^1$ au-dessus de la section nulle et est, de plus, $\phi_H$-invariant, c'est-à-dire que pour tous temps $t \in \mathbb{R}$, on a $\phi_H^t(\mathcal{L}) = \mathcal{L}$.*

### 4.4.2 Idée de la démonstration

Donnons maintenant les idées principales de la preuve du théorème 4.4.3. Considérons une sous-variété lagrangienne $\mathcal{L}$ de $T^*M$ qui est $H$-isotope à la section nulle, d'images $\mathcal{L}_t = \phi_H^t(\mathcal{L})$ par le flot Hamiltonien. Et supposons qu'il existe deux suites strictement croissantes d'entiers positifs $n_k$ et $m_k$ telles que $(\mathcal{L}_{n_k})_{k\geq 0}$ et $(\mathcal{L}_{-m_k})_{k\geq 0}$ qui convergent respectivement, et avec complexités réduites vers deux sous-variétés Lagrangiennes $\mathcal{L}_\alpha = \varphi_\alpha(0_{T^*M})$ et $\mathcal{L}_\omega = \varphi_\omega(0_{T^*M})$.

En suivant la démarche de la section 2.3.1, on construit des solutions variationnelles $u(t,q)$, $u_\alpha(t,q)$ et $u_\omega(t,q)$ de conditions initiales respectives, des sélecteurs de graphes $u(q)$, $u_\alpha(q)$ et $u_\omega(q)$ associés aux sous-variétés Lagrangiennes $\mathcal{L}$, $\mathcal{L}_\alpha$ et $\mathcal{L}_\omega$. Rappelons que ces solutions variationnelles sont accompagnées de familles $h_t$, $h_t^\alpha$ et $h_t^\omega$ de primitives de Liouvilles des sous-variétés $\mathcal{L}_t = \phi_H^t(\mathcal{L})$, $\mathcal{L}_t^\alpha = \phi_H^t(\mathcal{L}_\alpha)$ et $\mathcal{L}_t^\omega = \phi_H^t(\mathcal{L}_\omega)$ qui vérifient la relation 2.3.1.

La construction de Viterbo et ses résultats en topologie symplectique sur les invariants spectraux des fonctions génératrices, étudiés dans [Vit92], permettent d'établir l'inégalité suivante pour tout réel $a > 0$ :

$$\mathrm{Osc}\left(u(t+a,\cdot) - u_\alpha(a,\cdot)\right) \leq \mathrm{Osc}(l_t^\alpha) \quad \text{et} \quad \mathrm{Osc}\left(u(t+a,\cdot) - u_\omega(a,\cdot)\right) \leq \mathrm{Osc}(l_t^\omega) \qquad (4.4.3)$$

Ces inégalités permettent surtout de faire apparaître les oscillations des primitives de Liouville $\mathrm{Osc}(l_t)$ qui tendent à s'annuler 0, ce qui est l'une des hypothèses de la convergence à complexité bornée (Définition 4.4.1).

Nous avions mentionné dans la section 2.3.1 qu'une solution variationnelle associée à une condition initiale semi-convexe est alors une solution de viscosité de l'équation de Hamilton-Jacobi. Ceci n'est pas vérifié dans notre cas, mais nous devons pouvoir contourner cette difficulté afin d'arriver à montrer que $u$ est une solution (classique ou de viscosité) $C^1$-régulière de l'équation de Hamilton-Jacobi. Dans ce cas, le graphe de $du$ ne présentera plus de discontinuité comme dans la figure 2.5 et nous obtiendrons que $\mathscr{L}$ est le graphe de $du$.

L'idée est d'utiliser la proposition 2.1.2 sur la régularité des solutions de viscosité sur les courbes calibrées. Ceci permet de combiner à la fois les connaissance les propriétés des solutions variationnelles et les propriétés des solutions de viscosité qui devraient coïncider dans notre cadre d'étude.

Nous considérons le Lagrangien $L : \mathbb{T}^1 \times TM \to \mathbb{R}$, conjugué convexe du Hamiltonien de Tonelli $H$, et défini par

$$L(t, q, v) = \max_{p \in T_q^* M} \{p(v) - H(t, q, p)\}$$

Nous montrons en premier lieu que l'application $u$ est dominée par $L$, c'est-à-dire que pour toute courbe $\gamma : [a, b] \to M$, nous avons l'inégalité

$$u(b, \gamma(b)) - u(a, \gamma(a)) \leq \int_a^b L(\tau, \gamma(\tau), \dot{\gamma}(\tau))\, d\tau \tag{4.4.4}$$

Puis, nous remarquons que la proposition 2.1.2 reste valable pour les applications dominées. Ainsi, il ne reste plus qu'à montrer que pour tout $x = (q, p) \in \mathcal{L}$, la courbe $q(t) = \pi \circ \phi_L^t(x)$, où $\pi : TM \to M$ est la projection, est $u$-calibrée.

**Étude de la calibration.**

Pour toute application $L$-dominée $v$ et pour toute courbe $\gamma : [a, b] \to M$, nous définissons le défaut de calibration $\delta(v, \gamma)$ par

$$\delta(v, \gamma) = \int_a^b L(\tau, \gamma(\tau), \dot{\gamma}(\tau))\, d\tau - [v(b, \gamma(b)) - v(a, \gamma(a))] \geq 0 \tag{4.4.5}$$

Notre but est de montrer que pour tout $x = (q, p) \in \mathcal{L}$, l'orbite $x(t) = (q(t), p(t)) = \phi_H^t(x)$ vérifie que pour tous temps $a < b$, le défaut de calibration $\delta(u, q_{|[a,b]})$ est nul.

Nous remarquons d'abord que l'inégalité de Fenchel (2.0.3) et la relation 2.3.1 entre les primitives de Liouville résultent en l'identité

$$\int_a^b L(\tau, \gamma(\tau), \dot{\gamma}(\tau))\, d\tau = h_b(x(b)) - h_a(x(a))$$

Ainsi, le défaut de calibration se réduit à une comparaison entre les sélecteurs de graphes $u(t, \cdot)$ et les primitives de Liouville $h_t$ comme suit

$$\delta(u, q_{|[a,b]}) = [h_b(x(b)) - h_a(x(a))] - [u(b, q(b)) - u(a, q(a))] \tag{4.4.6}$$

Mais nous avons déjà établi dans (4.4.3) une comparaison de ce type seulement pour des temps $a$ et $b$ tendant vers $\pm\infty$. Il nous serait donc plus simple de commencer par montrer une calibration pour $u_\alpha$ et $u_\omega$ en utilisant l'hypothèse ii. sur le contrôle de dilatation des sous-variétés Lagrangiennes dans la définition 4.4.1 de la convergence à complexité réduite.

**Calibration des points limites.** Nous montrons que pour tout point limite $x_\omega = (q_\omega, p_\omega) \in \mathcal{L}_\omega$ de $x(n_k) = (q(n_k), p(n_k)) \in \mathcal{L}_{n_k}$, la courbe $q_\omega(t) = \pi \circ \phi_H^t(x_\omega)$ est calibrée par la solution variationnelle $u_\omega(t,q)$ associée à $\mathcal{L}_\omega$.

Dans cette preuve, nous montrons que si $u_k(t,q) = u(t+n_k, q)$ et $x_k(t) = (q_k(t), p_k(t)) = x(t+n_k)$, alors

$$\delta(u_\omega, q_{\omega|[a,b]}) = \lim_k \delta(u_k, q_{k|[a,b]})$$

Et nous intercalons $u_\omega$ et une primitive bien choisie $h_t^\omega$ de $\phi_H^t(\mathcal{L}_\omega)$ dans la formule (4.4.6) pour pouvoir comparer $u(t+n_k,\cdot) - u_\omega(t)$ à $h_t(x(t)) - h_t^\omega(x(t))$ en utilisant (4.4.3) et les hypothèses de contrôle des primitives dans la convergence à complexité réduite.
C'est dans cette étape que le contrôle de la primitive devient important, car c'est ce qui permet d'obtenir un défaut de calibration nul.

La même chose est montrée pour les temps négatifs, c'est-à-dire pour les courbes $q_\alpha(t)$ où $q_\alpha$ est un point limite de $q(-m_k)$.

Par ailleurs, nous montrons que cette calibration nous permet aussi d'étendre les formules (2.3.2) à ces points limites. Plus précisément, si on considère les primitives $h_0^\alpha(x) = \hbar_\alpha(0, -H(0,x), x)$ et $h_0^\omega(x) = \hbar_\omega(0, -H(0,x), x)$ de $\mathcal{L}_\alpha$ et $\mathcal{L}_\omega$, nous obtenons les égalités

$$u_\alpha(0, q_\alpha) = h_0^\alpha(x_\alpha) \quad \text{and} \quad u_\omega(0, q_\omega) = h_0^\omega(x_\omega) \tag{4.4.7}$$

**Calibration de tous les points.** Ensuite, nous montrons la calibration de la courbe $q(t)$ pour n'importe quel point $x = (q,p) \in \mathcal{L}$ comme suit. On fixe deux points limites $x_\alpha$ et $x_\omega$ des suites $x(-m_k)$ et $x(n_k)$. On a pour tous réels $a < b$, le défaut de calibration vérifie

$$\delta(u, q_{|[a,b]}) \le \liminf_k \delta(u, q_{|[-m_k, n_k]})$$

avec

$$\begin{aligned}\delta(u, q_{|[-m_k,n_k]}) &= [h_{n_k}(x_{n_k}) - h_{-m_k}(x_{-m_k})] - [u(n_k, q_{n_k}) - u(-m_k, q_{-m_k})] \\ &= [h_{n_k}(x_{n_k}) - u(n_k, q_{n_k})] - [h_{-m_k}(x_{-m_k}) - u(-m_k, q_{-m_k})]\end{aligned}$$

et avec des limites (à extraction près)

$$\lim_k h_{-m_k}(x_{-m_k}) - u(-m_k, q_{-m_k}) = h_0^\alpha(x_\alpha) - u_\alpha(0, q_\alpha) = 0$$
$$\lim_k h_{n_k}(x_{n_k}) - u(n_k, q_{n_k}) = h_0^\omega(x_\omega) - u_\omega(0, q_\omega) = 0$$

La positivité du défaut de calibration permet de conclure qu'il est nul. Et on conclue que la courbe $q(t)$ est calibrée par $u$.

Ainsi, nous déduisons de (2.1.9) que $p(t) = d_q u(t, q(t))$ et donc que $\mathcal{L}_t = \phi_H^t(\mathcal{L})$ est le graphe de $d_q u(t, \cdot)$ au-dessus de $M$.

## Deuxième partie

# Travaux menés durant la thèse
# Work Conducted During the PhD Thesis

# Chapitre 5

# Non-Autonomous Weak-KAM Theory

Let $M$ be a connected closed manifold. Denote by $TM$ its tangent bundle and by $T^*M$ its cotangent bundle with their respective projections $\pi_{TM} : TM \to M$ and $\pi_{T^*M} : T^*M \to M$. Unless it is ambiguous, these projections will be simply denoted by $\pi$. Denote by $\mathcal{C}(M, \mathbb{R})$ the set of continuous scalar maps on $M$ endowed with its usual infinite norm $\|.\|_\infty$ given by $\|u\|_\infty = \sup_{x \in M} |u(x)|$. The set $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ refers to the circle.

Let $H(t, x, p) : \mathbb{T}^1 \times T^*M \to \mathbb{R}$ be a 1-time periodic Tonelli Hamiltonian (See Definition 5.1.1) and let $\alpha_0$ be the Mañé critical value associated to $H$ (See Definition 5.1.3). We consider the *Hamilton-Jacobi equation*

$$\partial_t u + H(t, x, \partial_x u) = \alpha_0 \tag{5.0.1}$$

This equation has a well posed Cauchy problem in the viscosity sense (see [Lio82, CL83]). And A.Fathi [Fat97b] developed the weak-KAM theory which states that in the Tonelli framework, these solutions are generated by the Lax-Oleinik operator $\mathcal{T}^{s,t}$ defined on $\mathcal{C}(M, \mathbb{R})$ (see Definition 5.1.3). More specifically, for any real time $s \in \mathbb{R}$ and any $u_0 \in \mathcal{C}(M, \mathbb{R})$, there exists a unique viscosity solution $u(t, x) : [s, +\infty) \times M \to \mathbb{R}$ of (5.1.8) with $u(s, \cdot) = u_0$ defined by $u(t, x) = \mathcal{T}^{s,t} u_0(x)$.

Fathi initially developed his weak-KAM theory within the autonomous framework, focusing on weak-KAM solutions characterized as fixed points of the Lax-Oleinik operator. He proved that these solutions correspond to one-time periodic viscosity solutions. Through the Lax-Oleinik perspective, he demonstrated properties such as the existence of calibrated curves and the differentiability of weak-KAM solutions on these curves, along with the invariance of the differential graphs in negative times under the action of the Hamiltonian flow. These properties naturally linked to sets from the Aubry-Mather theory, where he established $C^{1,1}$ regularity on the projected Mather and Aubry sets. Building on this foundation, Fathi incorporated the Mañé perspective, viewing these sets in terms of global semi-static and static curves. This approach highlighted relationships between the Mather, Aubry, and Mañé sets, unveiling dynamical connections that govern the behavior of calibrated curves and, consequently, the evolution of weak-KAM solutions.

In this chapter, we introduce the preliminary tools from weak-KAM theory and Aubry-Mather theory that will be used in the subsequent chapters and throughout the results of this PhD thesis. To ensure thoroughness, we provide proofs for all key results, omitting only extensive details, thus offering an introduction to weak-KAM theory in the non-autonomous setting.

## 5.1 Tonelli Hamiltonians and the Lax-Oleinik Operator

These properties were initially established in the autonomous setting. And some were subsequently studied in the discrete case, which encompasses the non-autonomous setting (See [BB07, Zav10, Zav12]). A more specific exposition of the non-autonomous weak-KAM theory has been conducted in [Ber08] within the the pseudographs point of view. In this section, we provide a more classical presentation, similar to what has been written by A.Fathi in [Fat08].

### 5.1.1 Definitions

We define the notion of Tonelli Hamiltonians, initially introduced in [Mat91]. These are Hamiltonians that are convex and superlinear in the fibres, serving as the tame framework for the weak-KAM theory.

**Definition 5.1.1.** A 1-time-periodic Hamiltonian $H(t,x,p) : \mathbb{T}^1 \times T^*M \to \mathbb{R}$ is said *Tonelli* if it satisfies the following classical hypotheses :

— Regularity : $H$ is $\mathcal{C}^2$.

— Strict convexity : $\partial_{pp} H(t,x,p) > 0$ for all $(t,x,p) \in \mathbb{T}^1 \times T^*M$.

— Superlinearity : $H(t,x,p)/|p| \to \infty$ as $|p| \to \infty$ for each $(t,x) \in \mathbb{T}^1 \times M$.

— Completeness : The Hamiltonian vector field $X_H(t,x,p) = (\partial_p H(t,x,p), -\partial_x H(t,x,p))$ and hence its flow $\phi_H^{t,s}$ is complete in the sense that the flow curves are defined for all times $t \in \mathbb{R}$.

Under these assumptions, one can associate to $H(t,x,p)$ a time-periodic Tonelli *Lagrangian* $L(t,x,v) : \mathbb{T}^1 \times TM \to \mathbb{R}$ given by

$$L(t,x,v) = \max_{p \in T_x^*M} \{p(v) - H(t,x,p)\} \tag{5.1.1}$$

which symmetrically gives

$$H(t,x,p) = \max_{v \in T_xM} \{p(v) - L(t,x,v)\} \tag{5.1.2}$$

The *Euler-Lagrange flow* also named *Lagrangian flow* $\phi_L^{s,t}$ is the conjugate to the Hamiltonian flow $\phi_H^{s,t}$ by the *Legendre map*[1] $\mathcal{L}(t,x,v) = \big(t,x,\partial_v L(t,x,v)\big)$. We adopt the notation $\phi_L^t$ for $\phi_L^{0,t}$.

If $0 \leq s \leq t$ are two real times and $x$ and $y$ are two points of $M$, we define the following quantities

— For all absolutely continuous curve $\gamma : [s,t] \to M$, the *action* of $\gamma$ is

$$A_L(\gamma) = \int_s^t L(\tau,\gamma(\tau),\dot\gamma(\tau))\, d\tau \tag{5.1.3}$$

— the *potential* between $(s,x)$ and $(t,y)$ is

$$h_0^{s,t}(x,y) = \inf \left\{ A_L(\gamma) \;\middle|\; \begin{array}{rl} \gamma: & [s,t] \to M \\ & s \mapsto x \\ & t \mapsto y \end{array} \right\} \tag{5.1.4}$$

where the infimum is taken over such absolutely continuous curves $\gamma$.

**Remark 5.1.2.** The demanded absolute continuity of the curves enables to define this integral. All the curves and all the infimum over the curves that may be considered in this chapter will be in the family of absolutely continuous curves. We will refrain from recalling it every time.

We now introduce the Lax-Oleinik operator mentioned in the introduction and used to generate the viscosity solutions of the Hamilton-Jacobi equation (5.0.1).

**Definition 5.1.3.** Fix two times $s < t$.

1. The Tonelli assumptions imply that for all time $t \in \mathbb{R}$, the Legendre map $\mathcal{L}$ is a diffeomorphism between $\{t\} \times TM$ and $\{t\} \times T^*M$ (see [Fat08] for details).

1. The *Lax-Oleinik operator* $\mathcal{T}_0^{s,t} : \mathcal{C}(M,\mathbb{R}) \to \mathcal{C}(M,\mathbb{R})$ is defined as

$$\begin{aligned}\mathcal{T}_0^{s,t}u_0(x) &= \inf_{\substack{\gamma : [s,t] \to M \\ t \mapsto x}} \left\{ u_0(\gamma(s)) + \int_s^t L(\tau,\gamma(\tau),\dot{\gamma}(\tau))\, d\tau \right\} \\ &= \inf_{y \in M} \left\{ u_0(y) + h_0^{s,t}(y,x) \right\}\end{aligned} \tag{5.1.5}$$

We adopt the notations $\mathcal{T}_0^t$ for $\mathcal{T}_0^{0,t}$ and $\mathcal{T}_0$ for $\mathcal{T}_0^{0,1}$.

2. The *Mañé critical value* $\alpha_0$ is defined as

$$\alpha_0 = -\inf_{\mu} \left\{ \int_{\mathbb{T}^1 \times TM} L \, d\mu \right\} \tag{5.1.6}$$

where the infimum is taken over compact supported Borel probability measures $\mu$ invariant by the Euler-Lagrangian flow corresponding to $L$.

3. The *full Lax-Oleinik operator* $\mathcal{T}^{s,t} : \mathcal{C}(M,\mathbb{R}) \to \mathcal{C}(M,\mathbb{R})$ is defined as

$$\mathcal{T}^{s,t}u_0(x) = \mathcal{T}^{s,t}u_0(x) + \alpha_0.(t-s) \tag{5.1.7}$$

We adopt the notations $\mathcal{T}^t$ for $\mathcal{T}^{0,t}$ and $\mathcal{T}$ for $\mathcal{T}^{0,1}$.

One can verify that for all $s < t < \tau$, $\mathcal{T}^{t,\tau} \circ \mathcal{T}^{s,t} = \mathcal{T}^{s,\tau}$. Additionally, Since $L$ is time-periodic, $\mathcal{T}^{t+1} = \mathcal{T}^t \circ \mathcal{T}$. Hence $\{\mathcal{T}^n\}_{n\in\mathbb{N}}$ is a discrete semi-group acting on $\mathcal{C}(M,\mathbb{R})$, called the Lax-Oleinik semi-group. These properties are also verified by $\mathcal{T}_0$. The main focus of this chapter is the asymptotic behaviour of $\mathcal{T}$.

**Definition 5.1.4.**

1. To all scalar map $u \in \mathcal{C}(M,\mathbb{R})$ and all time $s \in \mathbb{R}$, we associate the *viscosity solution of the Hamilton-Jacobi equation* (5.0.1) $u : [s,+\infty) \times M \to \mathbb{R}$ defined by $u(t,x) = \mathcal{T}^{s,t}u(x)$.
2. If $\mathcal{T}u = u$, then $u$ is called a *weak-KAM solution* of the Hamilton-Jacobi equation (5.0.1). In other words, weak-KAM solutions are the initial data of one-time periodic viscosity solutions.
3. We denote by
   (a) $\mathrm{Fix}(\mathcal{T})$ the set of fixed points of the operator $\mathcal{T}$, namely the set of weak-KAM solutions.
   (b) $\mathrm{Per}(\mathcal{T})$ the set of periodic points of the operator $\mathcal{T}$.
   (c) For all $n \geq 1$, $\mathrm{Per}_n(\mathcal{T})$ the set of $n$-periodic points of the operator $\mathcal{T}$. The integer $n$ need not to be the minimal period of elements of $\mathrm{Per}_n(\mathcal{T})$.

**Remark 5.1.5.** 1. By abuse of language, we will often refer to the initial condition $u(x)$ as a viscosity solution while referring in reality to the corresponding viscosity solution $u(t,x) = \mathcal{T}^t u(x)$.

2. The Lax-Oleinik operator $\mathcal{T}_0^{s,t}$ is the generator of the viscosity solutions of the Hamilton-Jacobi equation
$$\partial_t u + H(t,x,d_x u) = 0 \tag{5.1.8}$$
It is easy to verify that a map $u \in \mathcal{C}(\mathbb{R}\times M,\mathbb{R})$ is a viscosity solution of (5.0.1) if and only if $u + \alpha_0 t$ is a viscosity solution of (5.1.8).

3. Replacing $H$ by $H - \alpha_0$ and $L$ by $L + \alpha_0$, it is always possible to assume that the Mañé critical value $\alpha_0$ is null. In this case, $\mathcal{T} = \mathcal{T}_0$.

### 5.1.2 Existence of Minimizing Curves

We give classical results on minimizing curves that stem directly from the theory of variational calculus. For certain standard statements, we will avoid providing tedious proofs that can be found in the autonomous framework in [Fat08] and [CI99].

**Definition 5.1.6.** A *minimizing curve* $\gamma : I \to M$ defined on an interval $I$ is a curve such that for all $s < t$ in $I$,
$$\int_s^t L(\tau,\gamma(\tau),\dot\gamma(\tau))\, d\tau = h_0^{s,t}(\gamma(s),\gamma(t)) \tag{5.1.9}$$

**Proposition 5.1.7.** *1. If for some times $s < t$, the curve $\gamma$ verifies (5.1.9), then it is minimizing on $[s,t]$.*

*2. A minimizing curve $\gamma$ is as regular as the Lagrangian L and it follows the Lagrangian flow $\phi_L$ i.e for all time $\tau \in [s,t]$, $(\gamma(\tau),\dot\gamma(\tau)) = \phi_L^{s,\tau}(\gamma(0),\dot\gamma(0))$. These curves verify the Euler-Lagrange equation*
$$\partial_x L(\tau,\gamma(\tau),\dot\gamma(\tau)) = \frac{d}{d\tau}\big(\partial_v L(\tau,\gamma(\tau),\dot\gamma(\tau))\big) \tag{5.1.10}$$

**Remark 5.1.8.** The first property is due to the fact that the quantity
$$\int_{s'}^{t'} L(\tau,\gamma(\tau),\dot\gamma(\tau)) - h_0^{s',t'}(\gamma(s'),\gamma(t'))\, d\tau \geq 0$$
is non-negative and is increasing with the size of the interval $[s',t']$.

A main brick in the study of Tonelli Lagrangians is the existence of minimizing curves.

**Theorem 5.1.9.** *(Tonelli's Theorem) Let $L : \mathbb{T}^1 \times TM \to \mathbb{R}$ be a Tonelli Lagrangian. Let $s < t$ be two real times and $x$ and $y$ be two points of $M$. Then, there exists a minimizing curve $\gamma : [s,t] \to M$ linking $x$ to $y$.*

A first application of this theorem to the Lax-Oleinik operator $\mathcal{T}$ gives the following

**Corollary 5.1.10.** *Let $u$ be a scalar map in $\mathcal{C}(M,\mathbb{R})$. For all times $s<t$ and for all point $x$ of $M$, there exists a minimizing curve $\gamma:[s,t]\to M$ such that*

$$\gamma(t)=x \quad \textit{and} \quad \mathcal{T}_0^{s,t}u(x)=u(\gamma(0))+A_L(\gamma) \tag{5.1.11}$$

*Proof.* We use the potential $h_0^{s,t}(\cdot,x)$ introduced in (5.1.4). Recall that

$$\mathcal{T}_0^{s,t}u(x)=\inf_{y\in M}\{u(y)+h_0^{s,t}(y,x)\}$$

We will see in Proposition 5.1.13 that the map $y\mapsto h_0^{s,t}(y,x)$ is continuous, and so is $y\mapsto u(y)+h_0^{s,t}(y,x)$ on the compact manifold $M$. Thus, it admits a minimizing point $y$. We get

$$\mathcal{T}_0^{s,t}u(x)=u(y)+h_0^{s,t}(y,x)$$

An application of Tonelli's theorem to $h_0^{s,t}(y,x)$ completes the proof. □

### 5.1.3 A Priori Compactness and Regularity

Another fundamental theorem is the A priori compactness property of minimizing curves for Tonelli Lagrangians. It has been first proven by John N. Mather in [Mat91].

**Theorem 5.1.11.** *(A Priori Compactness) Let $L:\mathbb{T}^1\times TM\to\mathbb{R}$ be a Tonelli Lagrangian and fix a small positive $\varepsilon>0$. Then, there exists a compact subset $K_\varepsilon$ of $TM$ such that every minimizing curve $\gamma:[s,t]\to M$ with $t-s\geq\varepsilon$ verifies $(\gamma(\tau),\dot\gamma(\tau))\in K_\varepsilon$.*

**Corollary 5.1.12.** *For fixed times $s<t$, if $\gamma_n:[s,t]\to M$ is a sequence of minimizing curves, then it admits a subsequence that $C^1$-converges to a minimizing curve $\gamma:[s,t]\to M$.*

*Proof.* By A priori compactness, we can extract a subsequence $(\gamma_{k_n}(s),\dot\gamma_{k_n}(s))$ that converges to $(x,v)\in TM$. Since the curves $\gamma_n$ are minimizing, we have for all $\tau\in[s,t]$, $(\gamma_n(\tau),\dot\gamma_n(\tau))=\phi_L^{s,\tau}(\gamma_n(s),\dot\gamma_n(s))$. Set $\gamma(\tau)=\pi\circ\phi_L^{s,\tau}(x,v)$ so that $(\gamma(\tau),\dot\gamma(\tau))=\phi_L^{s,\tau}(x,v)$. By continuity of the Lagrangian flow $\phi_L$, we deduce the $C^1$-convergence of the curves $\gamma_{k_n}$ to $\gamma$ on the time interval $[s,t]$. By continuity of $h_0$ obtained from proposition 5.1.13, we obtain

$$h_0^{s,t}(\gamma(s),\gamma(t))=\lim_n h_0^{s,t}(\gamma_{k_n}(s),\gamma_{k_n}(t))=\lim_n\int_s^t L(\tau,\gamma_{k_n}(\tau),\dot\gamma_{k_n}(\tau))\,d\tau=\int_s^t L(\tau,\gamma(\tau),\dot\gamma(\tau))\,d\tau$$

where the last limit is due to the $C^1$-convergence. We conclude the the curve $\gamma$ is minimizing. □

A consequence of the A Priori Compactness is the regularization property of the Lax-Oleinik operator $\mathcal{T}$.

**Proposition 5.1.13.** *For all positive $\varepsilon > 0$, there exists a positive constant $\kappa_\varepsilon > 0$ such that for all times $s < t$ with $t - s \geq \varepsilon$, we have*

1. *The potential $h_0^{s,t} : M \times M \to \mathbb{R}$ is $\kappa_\varepsilon$-Lipschitz. Moreover, we can take $\kappa_\varepsilon$ so that the time dependent potential $h_0 : \{0 \leq s \leq t - \varepsilon\} \times M \times M \to \mathbb{R}$ is still $\kappa_\varepsilon$-Lipschitz.*
2. *For all initial data $u \in \mathcal{C}(M, \mathbb{R})$, the maps $\mathcal{T}_0^{s,t} u$ and $\mathcal{T}^{s,t} u : M \to \mathbb{R}$ is $\kappa_\varepsilon$-Lipschitz on the set $\{0 \leq s \leq t - \varepsilon\} \times M$.*

Consequently, we get a regularity result on viscosity solutions.

**Corollary 5.1.14.** *For all viscosity solution $u : [s, +\infty) \times M \to \mathbb{R}$ and $v : \mathbb{R} \times M \to \mathbb{R}$ of the Hamilton-Jacobi equation (5.0.1), the families $(u(t,\cdot))_{t \geq s+1}$ and $(v(t,\cdot))_{t \in \mathbb{R}}$ are $\kappa_1$-equilipschitz.*

### 5.1.4 Calibrated Curves

Calibrated curves represent a type of minimizing curves that are well adapted to a given viscosity solution in the following sense.

**Definition 5.1.15.** Let $u(t,x)$ be a viscosity solution of (5.0.1). A curve $\gamma : I \subset \mathbb{R} \to M$ defined on a real interval $I$ is said *calibrated by $u$* or *$u$-calibrated* if for all times $s < t$ of $I$, we have

$$\begin{aligned} u(t, \gamma(t)) &= u(s, \gamma(s)) + \int_s^t L(\tau, \gamma(\tau), \dot{\gamma}(\tau))\, d\tau + \alpha_0.(t - s) \\ &= u(s, \gamma(s)) + h_0^{s,t}(\gamma(s), \gamma(t)) + \alpha_0.(t - s) \\ &= u(s, \gamma(s)) + h^{s,t}(\gamma(s), \gamma(t)) \end{aligned} \tag{5.1.12}$$

where the potential $h$ will be defined in Subsection 5.2.2.

**Remark 5.1.16.**
1. One observes from (5.1.12) that calibrated curves $\gamma$ realize the infimum in the definition (5.1.5) of the Lax-Oleinik operator. This means that all calibrated curves are minimizing and do follow the Lagrangian flow $\phi_L$.
2. Same as for minimizing curves in Remark 5.1.8, if a curve $\gamma$ verifies (5.1.12) for some times $s < t$, then it verifies it for all $s \leq s' < t' \leq t$ and $\gamma$ is calibrated by $u$ on the interval $[s, t]$.
3. For viscosity solutions of the translated Hamilton-Jacobi equation (5.1.8), the good equation of calibration is the following

$$\begin{aligned} u(t, \gamma(t)) &= u(s, \gamma(s)) + \int_s^t L(\tau, \gamma(\tau), \dot{\gamma}(\tau))\, d\tau \\ &= u(s, \gamma(s)) + h_0^{s,t}(\gamma(s), \gamma(t)) \end{aligned} \tag{5.1.13}$$

**Proposition 5.1.17.** *Let $u(t,x) : \mathbb{R}\times M \to \mathbb{R}$ be a viscosity solution of (5.0.1). For all points $x$ of $M$ and for all times $t \in \mathbb{R}$, $u$ admits a calibrated curve $\gamma_x : (-\infty, t] \to M$ with $\gamma(t) = x$.*

*Proof.* Fix a point $x$ in $M$ and a time $t$ in $\mathbb{R}$. We know that for all $s < t$, $u(t,\cdot) = \mathcal{T}^{s,t}u(s,\cdot)$. Then, applying Corollary 5.1.10, we obtain curves $\gamma_n : [-n,t] \to M$ with $\gamma_n(t) = x$, for large $n$, such that

$$u(t,\gamma_n(t)) = u(s,\gamma_n(-n)) + \int_{-n}^{t}\Big(L(\tau,\gamma_n(\tau),\dot{\gamma_n}(\tau)) + \alpha_0\Big)\,d\tau$$

The Remark 5.1.16 points out that these curves $\gamma_n$ are $u$-calibrated .
The A priori compactness Theorem 9.5.11 says that the curves $(\gamma_n, \dot{\gamma}_n)$ have their images in a same compact set $K$ of $TM$. Thus, for all compact interval $C$ of $(-\infty,t]$ and for large enough integer $n_0$, the family $(\gamma_{n|C})_{n\geq n_0}$ is relatively compact in the $C^1$-topology. Therefore, a diagonal arguments provides us with a curve $\gamma : (-\infty,t] \to M$ with $\gamma(t) = x$ such that the sequence $\gamma_n$ $C^1$-converges, up to extraction, to the curve $\gamma$ on all compact subsets of $(-\infty,t]$.
Let $s \leq t$ be a real time. For $n$ large enough, we have from the calibration of $\gamma_n$ that

$$u(t,\gamma_n(t)) = u(s,\gamma_n(s)) + \int_{s}^{t}\Big(L(\tau,\gamma_n(\tau),\dot{\gamma_n}(\tau)) + \alpha_0\Big)\,d\tau$$

And taking the limit on $n$, we get the calibration equation for $\gamma$.

$$u(t,\gamma(t)) = u(s,\gamma(s)) + \int_{s}^{t}\Big(L(\tau,\gamma(\tau),\dot{\gamma}(\tau)) + \alpha_0\Big)\,d\tau$$

□

One of the powerful results of the weak-KAM theory is the theorem of regularity on calibrated curves proved by A.Fathi in the autonomous case (see [Fat08]). For the sake of completeness, we provide a proof in the non-autonomous case following [AV17]. But first, we introduce a tool derived from convex analysis.

**Proposition 5.1.18.** *(Fenchel's inequality) For all $x$ in $M$ and all $(v,p) \in T_xM \times T_x^*M$*

$$p(v) \leq H(t,x,p) + L(t,x,v) \tag{5.1.14}$$

*with equality if and only if $p = \partial_v L(x,v)$ if and only if $v = \partial_p H(t,x,p)$.*

**Remark 5.1.19.** Note that if $\phi_H^t(x) = (x(t),p(t))$ is a curve that follows the Hamiltonian flow, then the Hamiltonian equations results in the equalities

$$\dot{x}(t) = \partial_p H\big(t,x(t),p(t)\big) \quad \text{and} \quad p(t) = \partial_v L\big(t,x(t),\dot{x}(t)\big) \tag{5.1.15}$$

**Theorem 5.1.20.** *Let $u$ be a viscosity solution of (5.1.8). If the curve $\gamma : I \to M$ is calibrated by $u$, then for all time $t$ in the interior of $I$, $u$ is differentiable at $(t,\gamma(t))$ with differential*

$$\partial_t u(t,\gamma(t)) = \alpha_0 - H(t,\gamma(t),d_x u(t,\gamma(t))) \quad \textit{and} \quad d_x u(t,\gamma(t)) = \partial_v L\big(t,\gamma(t),\dot\gamma(t)\big) \tag{5.1.16}$$

*Proof. Differentiability.* Let $\gamma : I \to M$ be a curve calibrated by $u$ and fix $(t,x) = (t,\gamma(t)) \in I \times M$. We will bound $u$ in a neighbourhood of $(t,x)$ by two $C^1$ maps that coincide with it at this point. Consider a chart $B_0 \subset M$ around $x$ and let $(s,y) \in \mathbb{R} \times B_0$ be close enough to $(t,x)$. Fix two reference times $t^+ < t < t^-$ in $I$ and $x^\pm = \gamma(t^\pm)$. From the calibration

$$u(t,x) = u(t^\pm,x^\pm) + \int_{t^\pm}^{t} \Big(L\big(\tau,\gamma(\tau),\dot\gamma(\tau)\big) + \alpha_0\Big)\, d\tau =: \psi^\pm(t,x) \tag{5.1.17}$$

and from the definition of viscosity solutions, we have

$$u(s,y) \le u(t^+,x^+) + \int_{t^+}^{s} \Big(L\big(\tau,\gamma^+_{(s,y)}(\tau),\dot\gamma^+_{(s,y)}(\tau)\big) + \alpha_0\Big)\, d\tau =: \psi^+(s,y) \tag{5.1.18}$$

and

$$u(s,y) \ge u(t^-,x^-) - \int_{s}^{t^-} \Big(L\big(\tau,\gamma^-_{(s,y)}(\tau),\dot\gamma^-_{(s,y)}(\tau)\big) + \alpha_0\Big)\, d\tau =: \psi^-(s,y) \tag{5.1.19}$$

where, in the chart $\mathbb{R} \times B_0$,

$$\gamma^\pm_{(s,y)}(\tau) = \gamma(\tau) + \frac{\tau - t^\pm}{s - t^\pm}(y - \gamma(s)) \tag{5.1.20}$$

are smooth families of curves linking $(t^\pm,x^\pm)$ to $(s,y)$ and such that $\gamma^\pm_{(t,x)} = \gamma$.
It is easy to see that $\psi^\pm$ are $C^1$. Moreover, $\psi^- \le u \le \psi^+$ with equalities at $(t,x)$. Then $u$ is differentiable at $(t,x)$.

*Evaluation of the Differential.* We differentiate (5.1.17) with respect to time $t$ without forgetting that $x = \gamma(t)$, to get

$$\partial_t u(t,\gamma(t)) + d_x u(t,\gamma(t)) = L\big(t,\gamma(t),\dot\gamma(t)\big) + \alpha_0 \tag{5.1.21}$$

And by Fenchel inequality (5.1.14) for $x = \gamma(t)$, $v = \dot\gamma(t)$ and $p = d_x u(t,\gamma(t))$, we have

$$0 = \partial_t u(t,\gamma(t)) + d_x u(t,\gamma(t)).\dot\gamma(t) - L\big(t,\gamma(t),\dot\gamma(t)\big) - \alpha_0 \le \partial_t u(t,x) + H(t,x,d_x u(t,x)) - \alpha_0 \tag{5.1.22}$$

We show the inverse inequality $\partial_t u(t,x) + H(t,x,d_x u(t,x)) - \alpha_0 \le 0$. Let $v$ be any element of $T_x M$ and let $\sigma : [t,t+1] :\to M$ be a curve such that $\sigma(0) = x$ and $\dot\sigma(0) = v$. Since $u$ is a viscosity solution, we have that for all $s \in [t, t+1]$,

$$\frac{u(s,\sigma(s)) - u(t,x)}{s-t} \leq \frac{1}{s-t} \int_t^s \Big( L(\tau, \sigma(\tau), \dot{\sigma}(\tau) + \alpha_0 \Big)\, d\tau$$

Letting $s$ tend to $t$, we deduce that

$$\partial_t u(t,x) + d_x u(t,\sigma(t)).v \leq L(t,x,v) + \alpha_0$$

Taking the supremum on $v \in T_x M$, we infer from the relation between $L$ and $H$ expressed in (5.1.2) the desired inequality $\partial_t u(t,x) + H(t,x,d_x u(t,x)) - \alpha_0 \leq 0$. Therefore, there is equality everywhere in (5.1.22), and in particular in the Fenchel inequality. Hence

$$\partial_t u(t,x) + H(t,x,d_x u(t,x)) = \alpha_0 \quad \text{and} \quad d_x u(t,\gamma(t)) = \partial_v L\big(t,\gamma(t),\dot{\gamma}(t)\big)$$

$\square$

**Remark 5.1.21.** This proposition can be refined as follows : the restriction of a viscosity solution $u$ is $C^{1,1}$-regular along the graph of a $u$-calibrated curve $\gamma : I \to M$, defined on an open interval $I$. A proof of this version will be presented in Proposition 9.5.8 within a different context.

## 5.2 The Aubry-Mather Sets

In this section, we introduce sets originating from Aubry-Mather theory, later extended through Mather's studies on minimizing measures [Mat91] and further developed in Fathi's weak-KAM theory [Fat08]. Additionally, we present Mañé's perspective, which connects these sets via essential dynamical properties [Mn97]. These sets—often explored in the context of action-minimizing measures and global dynamics—play a crucial role in understanding the structure and regularity of weak-KAM solutions, as well as in analyzing the stability and invariant properties within Hamiltonian dynamics.

### 5.2.1 The Mather Set

The Mather set, introduced by John N. Mather, is associated with the study of action-minimizing measures in Lagrangian and Hamiltonian systems. Some particular subsets specific to the non-autonomous case are defined as follows.

**Definition 5.2.1.** 1. A measure $\mu$ on $\mathbb{T}^1 \times TM$ is *a minimizing measure* if it is a Borel probability measure, invariant by the Euler-Lagrange flow $\phi_L$ and it satisfies

$$\int_{\mathbb{T}^1 \times TM} L \, d\mu = -\alpha_0 \tag{5.2.1}$$

where $\alpha_0$ is the Mañé critical value defined in (5.1.6) (and assumed to be null).

2. The *Mather set* $\tilde{\mathcal{M}}$ is defined by

$$\tilde{\mathcal{M}} = \bigcup_{\mu} \mathrm{Supp}(\mu) \subset \mathbb{T}^1 \times TM \tag{5.2.2}$$

where the union is on minimizing measures $\mu$.

3. The *projected Mather set* $\mathcal{M}$ is the projection of $\tilde{\mathcal{M}}$ to $\mathbb{T}^1 \times M$.

4. The *time-zero Mather set* $\tilde{\mathcal{M}}_0$ and its projected counterpart $\mathcal{M}_0$ are the intersections

$$\tilde{\mathcal{M}}_0 := \tilde{\mathcal{M}} \cap \big(\{0\} \times TM\big) \quad \text{and} \quad \mathcal{M}_0 := \mathcal{M} \cap \big(\{0\} \times M\big) \tag{5.2.3}$$

seen respectively as subsets of $TM$ and $M$.

5. We define the *recurrent Mather set* $\tilde{\mathcal{M}}_0^R \subset TM$ by

$$\tilde{\mathcal{M}}_0^R = \{\tilde{x} \in \tilde{\mathcal{M}}_0 \mid \tilde{x} \text{ is recurrent under the map } \phi_L^{-1}\} \tag{5.2.4}$$

and we denote its projection on $M$ by $\mathcal{M}_0^R$.

**Remark 5.2.2.** 1. More explicitly, the invariant measures $\mu$ featured in the definition are invariant by the maps $\Phi_L^\tau$, for all time $\tau > 0$, given by

$$\begin{aligned} \Phi_L^\tau : \mathbb{T}^1 \times TM &\longrightarrow \mathbb{T}^1 \times TM \\ (t, x, v) &\longmapsto (t + \tau, \phi_L^{t,t+\tau}(x, v)) \end{aligned} \tag{5.2.5}$$

2. We easily see from these definitions that the different Mather sets are invariant by either the Euler-Lagrange flow or its time-one map.

The next proposition has been proved by John N.Mather in the non-autonomous case. See proposition 4 of [Mat91]

**Proposition 5.2.3.** *The Mather set $\tilde{\mathcal{M}}$ is compact and non-empty and the recurrent Mather set $\tilde{\mathcal{M}}_0^R$ is dense in $\tilde{\mathcal{M}}_0$*

The following proposition has been proved by R.Mañé in [Mn97] and generalized by J-M.Roquejoffre and P.Bernard in [BR04] to non-wandering viscosity solutions, (see Proposition 7.2.1).

**Proposition 5.2.4.** *Let $\tilde{x}$ be an element of the Mather set $\tilde{\mathcal{M}}_0$ and $x = \pi(\tilde{x})$ be its projection in $\mathcal{M}_0$. Let $\gamma : \mathbb{R} \to M$ be the projection on $M$ of the Lagrangian orbit of $\tilde{x}$ i.e. $\gamma(t) = \pi \circ \phi_L^t(\tilde{x})$. Then, every weak-KAM solution $u$ is calibrated by $\gamma$.*

*Proof.* Let $u$ be a weak-KAM solution. Fix an integer $i \in \mathbb{N}$. We know from the definition of viscosity solutions that $\mathcal{T}^t u(0, \cdot) = u(t, \cdot)$. Thus, for all real time $t$ and all point $\tilde{y}$ of

$TM$, the definition of the operator $\mathcal{T}$ gives

$$u\big(t+k.i,\pi\circ\phi_L^{t,t+i}(\tilde{y})\big)-u\big(t,\pi\circ\tilde{y}\big)\leq A_L(\pi\circ\phi_L^{t,\tau}(\tilde{y}))=\int_t^{t+i}L(\tau,\phi_L^{t,\tau}(\tilde{y}))\,d\tau \tag{5.2.6}$$

Let $\mu$ be a minimizing measure on $\mathbb{T}^1\times TM$ that has $\tilde{x}$ in its support $\mathrm{Supp}(\mu)$. We integrate (5.2.6) in $(t,\tilde{y})\in[0,1]\times TM$ with respect to the lift measure $\mu$.

$$\begin{aligned}\int_0^1\int_{TM}u\big(t+i,\pi\circ\phi_L^{t,t+i}(\tilde{y})\big)\,d\mu-\int_0^1\int_{TM}u(t,\pi(\tilde{y}))\,d\mu&\\ \leq\int_0^1\int_{TM}\int_t^{t+i}L(\tau,\phi_L^{t,\tau}(\tilde{y}))\,d\tau\,d\mu&\end{aligned} \tag{5.2.7}$$

We start with the left-hand side of (5.2.7). We know from the definition of minimizing measures that $\mu$ is $\Phi_L^i$-invariant where $\Phi_L^i$ has been defined in (5.2.5). This writes

$$\begin{aligned}\int_0^1\int_{TM}u\big(t+i,\pi\circ\phi_L^{t,t+i}(\tilde{y})\big)\,d\mu&=\int_i^{i+1}\int_{TM}u(t,\pi(\tilde{y}))\,d\mu\\&=\int_0^1\int_{TM}u(t+i,\pi(\tilde{y}))\,d\mu\\&=\int_0^1\int_{TM}u(t,\pi(\tilde{y}))\,d\mu\end{aligned}$$

where we last used the one-time-periodicity of the weak-KAM solution $u$, resulting in the nullity of the left-hand side of (5.2.7).

A computation of remaining right-hand side of (5.2.7) gives

$$\begin{aligned}\int_0^1\int_{TM}\int_t^{t+i}L(\tau,\phi_L^{t,\tau}(\tilde{y}))\,d\tau\,d\mu&=\int_0^1\int_{TM}\int_0^{i}L(t+\tau,\phi_L^{t,t+\tau}(\tilde{y}))\,d\tau\,d\mu\\&=\int_0^i\int_0^1\int_{TM}L(t+\tau,\phi_L^{t,t+\tau}(\tilde{y}))\,d\mu\,d\tau\\&=\int_0^i\int_0^1\int_{TM}L(t,\tilde{y})\,d\mu\,d\tau\\&=i.\int_{\mathbb{T}^1\times TM}L(t,\tilde{y})\,d\mu=-\alpha_0=0\end{aligned} \tag{5.2.8}$$

where in the third line we used the $\Phi_L^{\tau}$-invariance of $\mu$, and in the fourth we used the time periodicity of the Lagrangian $L$.

We conclude that

$$\int_0^1\int_{TM}\left(\int_t^{t+i}L(\tau,\phi_L^{t,\tau}(\tilde{y}))\,d\tau-\Big[u\big(t+i,\pi\circ\phi_L^{t,t+i}(\tilde{y})\big)-u\big(t,\pi\circ\tilde{y}\big)\Big]\right)d\mu=0$$

where the integrand is non-negative. This implies that for $\mu$-almost all $(t,\tilde{y})$ in $\mathrm{Supp}(\mu)$, there is equality in the domination inequality (5.2.6). And by continuity of $L$ and $u$, the equality extends to $\mathrm{Supp}(\mu)$. Since $\mu$ is invariant by the Lagrangian flow $\phi_L$ and $\tilde{x}$ belongs to its support, we infer that the graph of the curve $(t,\gamma(t))$ also belongs to $\mathrm{Supp}(\mu)$.

Hence, for all integers $j$ and $i$,

$$u(j+i,\gamma(j+i)) = u(j,\gamma(j)) + \int_j^{j+i} L(\tau,\gamma(\tau),\dot\gamma(\tau))\,d\tau$$

Since $i$ and $j$ are arbitrary and thanks to inequality (5.2.6), the equality above holds for any real times $s<t$ which proves the calibration. □

**Remark 5.2.5.** We infer from Remark 5.1.16 that all curves in the Mather set $\mathcal{M}$ are minimizing.

### 5.2.2 The Potential $h$

For all times $s<t$ we define the potential $h^{s,t}: M\times M\to\mathbb{R}$ by

$$\begin{aligned} h^{s,t}(x,y) &= (t-s).\alpha_0 + h_0^{s,t}(x,y) \\ &= (t-s).\alpha_0 + \inf\left\{ A_L(\gamma) \;\middle|\; \begin{array}{rl} \gamma: & [s,t]\to M \\ & s\mapsto x \\ & t\mapsto y \end{array}\right\} \end{aligned} \tag{5.2.9}$$

And we adopt the notation $h^t$ for $h^{0,t}$.

**Remark 5.2.6.**
1. The Tonelli Theorem 5.1.9 states that the infimum in the definition above is always achieved by a minimizing curve.
2. When the Mañé critical value $\alpha_0$ is null, we get $h^{s,t} = h_0^{s,t}$.
3. We deduce from Proposition 5.1.13 the Lipschitz regularity of the potential $h^{s,t}$.

**Proposition 5.2.7.**
1. *(Triangular Inequality) For all real time $s<\tau<t$, and for all points $x$, $y$ and $z$ in $M$, we have the triangular inequality*

$$h^{s,t}(x,z) \le h^{s,\tau}(x,y) + h^{\tau,t}(y,z) \tag{5.2.10}$$

2. *For all $x\in M$, $h^t(x,\cdot)$ is a viscosity solution of the Hamilton-Jacobi equation (5.0.1) i.e. for all times $0<s<t$, $\mathcal{T}^{s,t}h^s(x,\cdot) = h^t(x,\cdot)$.*

*Proof.* 1. Let $\gamma_1:[s,\tau]\to M$ be a curve linking $x$ to $y$ and $\gamma_2:[\tau,t]\to M$ be a curve linking $y$ to $z$. And let $\gamma:[s,t]\to M$ be their concatenated curve linking $x$ to $z$. Then, we have

$$h^{s,t}(x,y) \le (t-s).\alpha_0 + A_L(\gamma) = (\tau-s).\alpha_0 + A_L(\gamma_1) + (t-\tau).\alpha_0 + A_L(\gamma_2)$$

and taking the infimum on the curves $\gamma_1$ and $\gamma_2$, we get the wanted inequality.

*2.* For $s < t \in \mathbb{R}$ and $x, y \in M$,

$$
\begin{aligned}
\mathcal{T}^{s,t} h^s(x,y) &= \inf_{\substack{\gamma_1 : [s,t] \to M \\ t \mapsto y}} \left\{ h^s\big(x, \gamma_1(s)\big) + A_L(\gamma_1) + (t-s).\alpha_0 \right\} \\
&= \inf_{\substack{\gamma_1 : [s,t] \to M \\ t \mapsto y}} \inf_{\substack{\gamma_2 : [0,s] \to M \\ 0 \mapsto x \\ s \mapsto \gamma_1(s)}} \{ s.\alpha_0 + A_L(\gamma_2) + A_L(\gamma_1) + (t-s).\alpha_0 \} \\
&= \inf_{\substack{\gamma : [0,t] \to M \\ 0 \mapsto x \\ t \mapsto y}} \{ t.\alpha_0 + A_L(\gamma) \} = h^t(x,y)
\end{aligned}
$$

□

We will show that this potential $h$ is bounded for $t - s > \varepsilon$ for some fixed $\varepsilon$. To achieve this, we need to consider its lower and upper bounds.

**Definition 5.2.8.** We define the maps $m$ and $M : M \times M \to \mathbb{R}$ as

$$m(x,y) = \inf_{n \geq 1} h^n(x,y) \quad \text{and} \quad M(x,y) = \sup_{n \geq 1} h^n(x,y) \tag{5.2.11}$$

The associated time-dependant maps are defined as

$$m^t(x,y) = m(t,x,y) = \inf_{n \geq 1} h^{n+t}(x,y) \quad \text{and} \quad M^t(x,y) = M(t,x,y) = \sup_{n \geq 1} h^{t+n}(x,y) \tag{5.2.12}$$

And more generally, for every two times $s < t$, we define

$$m^{s,t}(x,y) = \inf_{n \geq 1} h^{s,n+t}(x,y) \quad \text{and} \quad M^{s,t}(x,y) = \sup_{n \geq 1} h^{s,t+n}(x,y) \tag{5.2.13}$$

**Remark 5.2.9.** There are some subtleties that justify taking the infimum over $n \geq 1$ while excluding $n = 0$. However, for the purpose of the following proposition, the chosen definition is sufficient.

**Proposition 5.2.10.** *For all $(t,x,y) \in [0,+\infty) \times M \times M$, the maps $m(t,x,y)$ and $M(t,x,y)$ are finite, and at $t = 0$ we have the uniform bound*

$$\max(\|m\|_\infty, \|M\|_\infty) \leq 2\kappa_1 . \operatorname{diam}(M) \tag{5.2.14}$$

*where $\kappa_1$ is the Lipschitz constant introduced in Proposition 5.1.13.*

*Proof.* The proof of this proposition is divided into two main steps. The first step is to find points $x_n$ and $y_n$ such that $\frac{1}{n}h^n(x_n, y_n)$ converges to a finite limit using the ergodic decomposition and the A Priori Compactness Theorem 9.5.11. The second step is to estimate the oscillation of $h$ with respect to this established finite limit using the Lipschitz regularity of the potential 5.1.13.

*Step 1.* Let $\mu$ be a minimizing measure. We infer from proposition 5.2.3 that $supp(\mu)$ is compact. We will need the ergodic decomposition of invariant measures. For that purpose, we introduce some definitions following [Mn87].

- Let $\Sigma^0$ be the set of points $(t, x, v) \in \mathrm{Supp}(\mu)$ such that for every continuous map $\theta : \mathbb{T}^1 \times TM \to \mathbb{R}$, the limit

$$\lim_{n\to+\infty} \frac{1}{n} \int_0^n \theta(t+\tau, \phi_L^{t,t+\tau}(x, v))\, d\tau$$

  exists and is finite.

- For all points $(t, x, v)$ of $\Sigma^0$, we define the invariant Borel measure $\mu_{(t,x,v)}$ defined with the Riesz's representation theorem as

$$\begin{array}{rccc} \mu_{(t,x,v)}: & \mathcal{C}(\mathbb{T}^1 \times TM, \mathbb{R}) & \longrightarrow & \mathbb{R} \\ & \theta & \longmapsto & \lim\limits_{n\to+\infty} \frac{1}{n}\int_0^n \theta(t+\tau, \phi_L^{t,t+\tau}(x, v))\, d\tau \end{array} \tag{5.2.15}$$

- Let $\Sigma$ be the set of points $(t, x, v)$ of $\Sigma^0$ such that $\mu_{(t,x,v)}$ is ergodic and $(t, x, v) \in \mathrm{Supp}(\mu_{(t,x,v)})$.

Then, we have the following theorem

**Theorem 5.2.11** (Ergodic Decomposition of Invariant Measures, [Mn87])**.** *The set $\Sigma$ is of full measure i.e $\mu(\Sigma) = 1$, and for all map $\theta \in \mathcal{C}(\mathbb{T}^1 \times TM, \mathbb{R})$ we have*

$$\int_{\mathbb{T}^1\times TM} \left( \int_{\mathbb{T}^1\times TM} \theta \, d\mu_{(t,x,v)} \right) d\mu = \int_{\mathbb{T}^1\times TM} \theta \, d\mu \tag{5.2.16}$$

We apply the theorem with $\theta$ being the Lagrangian $L$. Recall that the measure $\mu$ is chosen to be minimizing, hence we get

$$-\alpha_0 = \int_{\mathbb{T}^1\times TM} L\, d\mu = \int_{\mathbb{T}^1\times TM} \left( \int_{\mathbb{T}^1\times TM} L\, d\mu_{(t,x,v)} \right) d\mu \tag{5.2.17}$$

Moreover, we know from the definition (5.1.6) of the Mañé critical value $\alpha_0$ that for all $(t, x, v) \in \Sigma^0$,

$$-\alpha_0 \le \int_{\mathbb{T}^1\times TM} L\, d\mu_{(t,x,v)} \tag{5.2.18}$$

Hence, we deduce from (5.2.17) that for $\mu$-almost all $(t, x, v) \in \Sigma^0$, the measure $\mu_{(t,x,v)}$ is minimizing and we have equality in (5.2.18).

Let $(t, x', v')$ be such a point and consider $(0, x, v) = (\Phi_L^t)^{-1}(t, x', v')$ where the map $\Phi_L$ has been defined in (5.2.5). Let us show that $\mu_{(t,x',v')} = \mu_{(0,x,v)}$. For all integer $n \geq 0$ and scalar map $\theta \in \mathcal{C}(\mathbb{T}^1 \times TM, \mathbb{R})$, we have

$$\begin{aligned}
\frac{1}{n}\int_0^n \theta(\tau, \phi_L^\tau(x,v))\, d\tau &= \frac{1}{n}\int_{-t}^{n-t} \theta(t+\tau, \phi_L^{t+\tau}(x,v))\, d\tau \\
&= \frac{1}{n}\int_{-t}^{n-t} \theta(t+\tau, \phi_L^{t,t+\tau}(x',v'))\, d\tau \\
&= \frac{1}{n}\Big[\int_{-t}^{0} \theta(t+\tau, \phi_L^{t,t+\tau}(x',v'))\, d\tau + \int_0^{\lfloor n-t\rfloor} \theta(t+\tau, \phi_L^{t,t+\tau}(x',v'))\, d\tau \\
&+ \int_{\lfloor n-t\rfloor}^{n-t} \theta(t+\tau, \phi_L^{t,t+\tau}(x',v'))\, d\tau\Big]
\end{aligned}$$

where $\lfloor\cdot\rfloor$ stands for the floor map, and with

$$\begin{aligned}
\left|\frac{1}{n}\int_{-t}^{0} \theta(t+\tau, \phi_L^{t,t+\tau}(x',v'))\, d\tau\right| &\leq \frac{|t|}{n}\left\|\theta_{|\operatorname{Supp}(\mu)}\right\|_\infty \longrightarrow 0 \quad \text{as } n \to +\infty \\
\left|\frac{1}{n}\int_{\lfloor n-t\rfloor}^{n-t} \theta(t+\tau, \phi_L^{t,t+\tau}(x',v'))\, d\tau\right| &\leq \frac{1}{n}\left\|\theta_{|\operatorname{Supp}(\mu)}\right\|_\infty \longrightarrow 0 \quad \text{as } n \to +\infty
\end{aligned}$$

Thus, we deduce that

$$\begin{aligned}
\int_{\mathbb{T}^1\times TM} \theta\, d\mu_{(0,x,v)} &= \lim_{n\to+\infty}\frac{1}{n}\int_0^n \theta(\tau, \phi_L^\tau(x,v))\, d\tau \\
&= \lim_{n\to+\infty}\frac{1}{n}\int_0^{\lfloor n-t\rfloor} \theta(t+\tau, \phi_L^{t,t+\tau}(x',v'))\, d\tau \\
&= \lim_{n\to+\infty}\frac{1}{\lfloor n-t\rfloor}\int_0^{\lfloor n-t\rfloor} \theta(t+\tau, \phi_L^{t,t+\tau}(x',v'))\, d\tau \\
&= \int_{\mathbb{T}^1\times TM} \theta\, d\mu_{(t,x',v')}
\end{aligned}$$

Therefore, the measure $\mu_{(0,x,v)}$ is also minimizing and we obtain

$$\lim_{n\to+\infty}\frac{1}{n}\int_0^n L(\tau, \phi_L^\tau(x,v))\, d\tau = \int_{\mathbb{T}^1\times TM} L\, d\mu_{(0,x,v)} = -\alpha_0 \tag{5.2.19}$$

Let $x_n = \pi \circ \phi_L^n(x,v)$. We claim that the limit above implies

$$\lim_n \frac{1}{n} h_0^n(x, x_n) = -\alpha_0 \tag{5.2.20}$$

In fact, if $\gamma_n : [0,n] \to M$ is a curve realizing $h_0^n(x,x_n)$, we consider the measure $\nu_n$ represented by $\theta \mapsto \frac{1}{n}\int_0^n \theta(\tau, \gamma(\tau), \dot\gamma(\tau))d\tau$ which is supported on $(\gamma_n, \dot\gamma_n)$. Let $\nu_{k_n}$ be any subsequence of $\nu_n$. By the A priori compactness Theorem 9.5.11, a subsequence of $\nu_{k_n}$ weakly converges to an invariant, compactly supported Borel measure $\nu$. We obtain a subsequence $k'_n$ of $k_n$ such that

$$\lim_n \frac{1}{k'_n} h_0^{k'_n}(x, x_{k'_n}) = \int_{\mathbb{T}^1 \times TM} L \, d\nu \geq -\alpha_0$$

where the final inequality comes from definition of the Mañé critical value $\alpha_0$. However, we know from (5.2.19) that

$$\lim_n \frac{1}{k'_n} h_0^{k'_n}(x, x_{k'_n}) \leq \lim_n \frac{1}{k'_n} \int_0^{k'_n} L(\tau, \phi_L^\tau(x, v)) \, d\tau = -\alpha_0$$

We get a double inequality, and thus equality of the limits. Moreover, this procedure shows that $-\alpha_0$ is the only limit value of the sequence $\frac{1}{n} h_0^n(x, x_n)$, which proves (5.2.20).

*Step 2.* Now let $M_n = \max\limits_{M \times M} h_0^n$ and $m_n = \min\limits_{M \times M} h_0^n$. We have for all integers $n, m \geq 1$ and points $y$ and $z$ in $M$,

$$h_0^{m+n}(y, z) \leq h_0^m(y, y) + h_0^n(y, z) \leq M_m + h_0^n(y, z)$$

And taking the maximum on $M \times M$, we obtain the subadditive inequality

$$M_{n+m} \leq M_m + M_n$$

and the sequence $M_n$ is subadditive. A classical consequence of this is that there exist $\beta \in \mathbb{R} \cup \{-\infty\}$ such that

$$\lim_n \frac{M_n}{n} = \inf_{n \geq 0} \frac{M_n}{n} = \beta \tag{5.2.21}$$

Similarly, the sequence $-m_n$ is subadditive and there exist $\beta' \in \mathbb{R} \cup \{+\infty\}$ such that

$$\lim_n \frac{m_n}{n} = \sup_{n \geq 0} \frac{m_n}{n} = \beta' \tag{5.2.22}$$

Additionally, we know from the regularity Proposition 5.1.13 on the potential $h_0$ that for all $n \geq 1$,

$$0 \leq M_n - m_n \leq \kappa_1 . 2 \operatorname{diam} M \tag{5.2.23}$$

where $\operatorname{diam} M := \max\{d(x, y), (x, y) \in M \times M\}$ is the diameter of $M$. This uniform bound immediately implies that $\beta = \beta' \in \mathbb{R}$. Considering the points $x$ and $x_n$ of the established limit (5.2.20), we get

$$\frac{m_n}{n} \leq \frac{1}{n} h^n(x, x_n) \leq \frac{M_n}{n}$$

and taking the limit on $n$, we infer the equality $\beta = \beta' = -\alpha_0$.

By gathering (5.2.21), (5.2.22) and (5.2.23), we deduce that

$$-\kappa_1.2\operatorname{diam} M - \alpha_0.n \leq m_n \leq -\alpha_0.n \leq M_n \leq \kappa_1.2\operatorname{diam} M - \alpha_0.n$$

leading to

$$-\kappa_1.2\operatorname{diam} M \leq \min_{M\times M} h^n = m_n + \alpha_0.n \leq 0 \leq \max_{M\times M} h^n = M_n + \alpha_0.n \leq \kappa_1.2\operatorname{diam} M$$

Therefore, we obtain the desired bounding on the maps $m$ and $M$. The time continuity of $h$ extends the (uniform) finiteness to $m(t,x,y)$ and $M(t,x,y)$. □

**Remark 5.2.12.** Note from the previous proof that the Mañé critical value $\alpha_0$ is finite.

### 5.2.3 The Peierls Barrier and the Peierls Set

The Peierls barrier, introduced by Mather in [Mat93], provides various viscosity solutions to the Hamilton-Jacobi equation. These solutions are crucial for describing all weak-KAM solutions (See [Con01, CISM13]). In this section, we introduce the Peierls barrier and compile some of its key properties.

**Definition 5.2.13.** The *Peierls barrier* $h^\infty : M^2 \to \mathbb{R}$ is defined as

$$h^\infty(x,y) = \liminf_{n\to\infty} h^n(x,y) \tag{5.2.24}$$

Its time dependence is defined as follows

$$h^\infty(t,x,y) = h^{\infty+t}(x,y) = \liminf_{n\to\infty} h^{n+t}(x,y) \tag{5.2.25}$$

More generally, for every two times $s$ and $t$, we define

$$h^{s,\infty+t}(x,y) = \liminf_{n\to\infty} h^{s,n+t}(x,y) \tag{5.2.26}$$

**Proposition 5.2.14.** 1. *(Finiteness) For all $(t,x,y) \in \mathbb{R}\times M\times M$, the Peierls barrier $h^\infty(t,x,y)$ is finite.*

2. *(Regularity) The Peierls barrier $h^\infty$ is $\kappa_\varepsilon$-Lipschitz for all $\varepsilon > 0$ with $\kappa_\varepsilon$ being the same Lipschitz value introduced in Proposition 5.1.13.*

3. *(Weak-KAM Solution) For all $x \in M$, $h^\infty(\cdot,x,\cdot)$ is a weak-KAM solution of the Hamilton-Jacobi equation (5.0.1).*

4. *(Liminf Property) For all points $x$ and $y$ in $M$ and all sequences of points $(x_n)_n$ and $(y_n)_n$ respectively converging to $x$ and $y$, we have*

$$h^\infty(x,y) = \liminf_{n} h^n(x_n,y_n) \tag{5.2.27}$$

5. *(Triangular Inequality) For all point $x$, $y$ and $z$ in $M$, we have the triangular inequality*

$$h^{\infty}(x,z) \leq h^{\infty}(x,y) + h^{\infty}(y,z) \tag{5.2.28}$$

6. *And for all times $s < \tau < t$, we have the triangular inequalities*

$$\begin{aligned} h^{s,\infty+t}(x,z) &\leq h^{s,\infty+\tau}(x,y) + h^{\tau,t}(y,z) \\ h^{s,\infty+t}(x,z) &\leq h^{s,\tau}(x,y) + h^{\tau,\infty+t}(y,z) \end{aligned} \tag{5.2.29}$$

7. *(Non-negativity) For every point $x$ in $M$, we have $h^{\infty}(x,x) \geq 0$.*

*Proof.* 1. Direct consequence of Proposition 5.2.10.

2. Direct consequence of Proposition 5.1.13.

3. For all $x \in M$, we know from Proposition 5.2.7 that $h^n(\cdot, x, \cdot)$ is a viscosity solution. Moreover, we will see in Proposition 6.2.3 that in the Tonelli framework, viscosity solutions are stable under the liminf operation, which yields that $h^{\infty}(\cdot, x, \cdot) = \liminf_n h^n(\cdot, x, \cdot)$ is a viscosity solution. The one-time periodicity implies that it is also a weak-KAM solution and $\mathcal{T} h^{\infty}(x, \cdot) = h^{\infty+1}(x, \cdot) = h^{\infty}(x, \cdot)$.

Note that it will be the weak-KAM solution used to prove Proposition 6.2.4 on the existence of weak-KAM solutions.

4. We have from the regularity of $h$ that for all integers $n$ and $k \geq 1$

$$|h^k(x,y) - h^k(x_n, y_n)| \leq \kappa_1.\big(d(x,x_n) + d(y,y_n)\big)$$

Hence, setting $k = n$ and taking the liminf on $n$, we get

$$h^{\infty}(x,y) = \liminf_n h^n(x,y) = \liminf_n h^n(x_n, y_n)$$

5. Fix three points $x$, $y$ and $z$ in $M$. Let $k_n$ and $q_n$ be two increasing sequences of integers such that $h^{\infty}(x,y) = \lim_n h^{k_n}(x,y)$ and $h^{\infty}(y,z) = \lim_n h^{q_n}(y,z)$. Then, we have

$$\begin{aligned} h^{\infty}(x,z) = \liminf_{n\to\infty} h^n(x,y) &\leq \liminf_n h^{k_n+q_n}(x,y) \\ &\leq \liminf_n h^{k_n}(x,y) + h^{q_n}(y,z) \\ &= \lim_n h^{k_n}(x,y) + h^{q_n}(y,z) \\ &= h^{\infty}(x,y) + h^{\infty}(y,z) \end{aligned}$$

where we used the triangular inequality (5.2.10) in the second line.

6. As the previous point, we fix three points $x$, $y$ and $z$ in $M$ and three times $s$ and $\tau < t$. Let $k_n$ be an increasing sequences of integers such that $h^{s,\infty+\tau}(x,y) = \lim_n h^{s,k_n+\tau}(x,y)$. Then we have,

$$\begin{aligned} h^{s,\infty+t}(x,z) &= \liminf_{n\to\infty} h^{s,n+t}(x,y) \leq \liminf_{n} h^{s,k_n+t}(x,y) \\ &\leq \liminf_{n} h^{s,k_n+\tau}(x,y) + h^{k_n+\tau,k_n+t}(y,z) \\ &= h^{s,\infty+\tau}(x,y) + h^{\tau,t}(y,z) \end{aligned}$$

The second inequality is analogous.

7. Consider a weak-KAM solution $u$. Then, for all point $x$ in M and for all integer $n \geq 0$, we get from the definition of $u$ that

$$0 = u(x) - u(x) = u(n,x) - u(0,x) \leq h^n(x,x)$$

Taking the liminf on $n$, we get the inequality $h^\infty(x,x) \geq 0$. □

We finally introduce the Peierls set, also known as the projected (time zero) Aubry set as follows

**Definition 5.2.15.** We define the *Peierls set* $\mathcal{A}_0$ in $M$ as follows

$$\mathcal{A}_0 = \{x \in M \mid h^\infty(x,x) = 0\} \tag{5.2.30}$$

**Proposition 5.2.16.** *For all $x \in \mathcal{M}_0$, we have $h^\infty(x,x) = 0$. In other words, we have the inclusion of sets $\mathcal{M}_0 \subset \mathcal{A}_0$*

*Proof.* Let $x$ be in $\mathcal{M}_0^R$ with lift $\tilde{x}$ in $\tilde{\mathcal{M}}$ and set $x(t) = \pi \circ \phi_L^t(\tilde{x})$. Let $k_n$ be an increasing sequence of integers such that $x(-k_n)$ converge to $x$. From Proposition 5.2.4, we have that the curve $x(t) = \pi \circ \phi_L^t(x,v)$ is calibrated by any weak-KAM solution $u$ and

$$h^{k_n}(x(-k_n),x) = u(x) - u(-k_n, x(-k_n)) = u(x) - u(x(-k_n)) \longrightarrow 0 \quad \text{as } n \to \infty$$

where we use the continuity of $u$. Thus, by the liminf Property (7.3.5), and by the non-negativity Property 7 of Proposition 5.2.14, we obtain

$$0 \leq h^\infty(x,x) \leq h^{\underline{k}}(x,x) = \liminf_{n} h^{k_n}(x(-k_n),x) = 0$$

We proved that $h^\infty(x,x) = 0$ on $\mathcal{M}_0^R$. Additionally, it was stated in Proposition 5.2.3 that this set is dense in the Mather set $\mathcal{M}_0$. Therefore, we deduce by continuity that $h^\infty(x,x) = 0$ on $\mathcal{M}_0$. □

### 5.2.4 The Mañé Set

In this subsection, we present Mañé 's perspective on Aubry-Mather theory, focusing on absolutely minimizing curves, classified as static and semi-static curves. We will define the Mañé set, which holds particular significance in weak-KAM theory. This set will be instrumental in proving the regularity of the recurrent viscosity solution constructed in Section 8.3.

The definition of the Mañé Set requires the introduction of a new potential.

**Definition 5.2.17.** 1. The *Mañé potential* $m : M \times M \to \mathbb{R}$ is defined by

$$m(x,y) = \inf_{n\geq 0} h^n(x,y) \tag{5.2.31}$$

with a time evolution counterpart $m : \{(s,t) \in \mathbb{T}^1 \times \mathbb{T}^1 \mid s \leq t\} \times M \times M \to \mathbb{R}$ defined by

$$m^{s,t}(x,y) = \inf_{n\geq 0} h^{s,t+n}(x,y) \tag{5.2.32}$$

We use the notation $m^t$ for $m^{0,t}$.

2. A curve $\gamma : I \to \mathbb{R}$ is said *semi-static* if for all times $s < t$ in $I$, we have

$$m^{s,t}(\gamma(s),\gamma(t)) = \alpha_0.(t-s) + \int_s^t L(\tau,\gamma(\tau),\dot\gamma(\tau))\, d\tau \tag{5.2.33}$$

3. The *Mañé set* $\tilde{\mathcal{N}}$ is the subset of $\mathbb{T}^1 \times TM$ defined as

$$\tilde{\mathcal{N}} = \{(t,\gamma(t),\dot\gamma(t)) \mid \gamma : \mathbb{R} \to M \text{ is a semi-static curve }\} \tag{5.2.34}$$

**Remark 5.2.18.** 1. The first term of the infimum is the quantity $h^0$ defined as

$$h^0(x,y) = \begin{cases} 0 & \text{if } x = y \\ +\infty & \text{if } x \neq y \end{cases} \tag{5.2.35}$$

It follows that for any point $x$ of $M$, $m(x,x) = 0$. This also implies that the map $m$ can exhibit discontinuity at the diagonal of $M \times M$.

2. Away from the diagonal of $M\times M$, the Mañé potential $m^{s,t}$ is continuous on its time and space variables.

3. Note that a semi-static curve is minimizing as for any times $s < t$,

$$A_{L+\alpha_0}(\gamma) = m^{s,t}(\gamma(s),\gamma(t)) \leq h^{s,t}(\gamma(s),\gamma(t)) \leq A_{L+\alpha_0}(\gamma)$$

and we get equality everywhere.

4. Analogously to Remarks 5.1.8 and 5.1.16, if for fixed times $s < t$ the identity (5.2.33) holds, then the curve $\gamma$ is semi-static on $[s,t]$.

The regularity Proposition 5.1.13 for the finite-time potential $h^t$ is reflected on the regularity of the Mañé potential $m^{s,t}$ as stated below.

**Proposition 5.2.19.** *For all positive real number $\varepsilon > 0$ and all times $s < t + \varepsilon$, the Mañé potential $m^{s,t}$ is $\kappa_\varepsilon$-Lipschitz with $\kappa_\varepsilon$ being the same Lipschitz value figuring in Proposition 5.1.13.*

Below is another characterization of semi-static curves. Only an implication is stated here, but the inverse statement will be shown in Proposition 5.2.26.

**Proposition 5.2.20.** *Let $u$ be a weak-KAM solution of (5.0.1) and let $\gamma : I \to M$ be a calibrated curve by $u$. Then the curve $\gamma$ is semi-static.*

*Proof.* Let $s < t$ be two times in $I$. We have by definition of weak-KAM solutions that for all integer $k$ such that $s \leq t + k$

$$\begin{aligned} u(t,\gamma(t)) = u(t+k,\gamma(t)) + k.\alpha_0 &= \mathcal{T}^{s,t+k}u(s,\gamma(t)) + k.\alpha_0 \\ &\leq u(s,\gamma(s)) + h_0^{s,t+k}(\gamma(s),\gamma(t)) + k.\alpha_0 \\ &= u(s,\gamma(s)) + h^{s,t+k}(\gamma(s),\gamma(t)) - \alpha_0.(t-s) \end{aligned}$$

so that

$$A_{L+\alpha_0}(\gamma_{|[s,t]}) = A_L(\gamma_{|[s,t]}) + \alpha_0.(t-s) = u(t,\gamma(t)) - u(s,\gamma(t)) + \alpha_0.(t-s) \leq h^{s,t+k}(\gamma(s),\gamma(t))$$

And taking the infimum over the integers $k$, we get that $A_{L+\alpha_0}(\gamma_{|[s,t]}) = m^{s,t}(\gamma(s),\gamma(t))$ meaning that $\gamma$ is a semi-static curve. ☐

**Proposition 5.2.21.** *The following inclusion holds*

$$\tilde{\mathcal{M}} \subset \tilde{\mathcal{N}} \tag{5.2.36}$$

*Proof.* Let $(s,\tilde{x})$ be an element of $\tilde{\mathcal{M}}$ and $\gamma : \mathbb{R} \to M$ be the curve of $M$ defines as $\gamma(t) = \pi \circ \phi_L^{s,t}(\tilde{x})$. We need to show that $\gamma$ is semi-static.

Fix a weak-KAM solution $u$. We know from Proposition 5.2.4 that $\gamma$ is calibrated by $u$. Thus, for all integers $q < p$ and $k \geq 1$, and for all curves $\sigma : [q, q+k] \to M$ linking $\gamma(q)$

to $\gamma(p)$, we have

$$\begin{aligned}A_{L+\alpha_0}(\gamma_{|[q,p]}) &= A_L(\gamma_{|[q,p]}) + \alpha_0.(p-q)\\ &= u(p,\gamma(p)) - u(q,\gamma(q)) + \alpha_0.(p-q)\\ &= u(q+k,\gamma(p)) + \alpha_0.(q+k-p) - u(q,\gamma(q)) + \alpha_0.(p-q)\\ &= u(q+k,\gamma(p)) - u(q,\gamma(q)) + \alpha_0.k\\ &\leq A_L(\sigma) + \alpha_0.k = A_{L+\alpha_0}(\sigma)\end{aligned}$$

where we used the definition of weak-KAM solutions in the second line. Taking the infimum over such curves $\sigma$ while varying $k$, this implies that

$$A_L(\gamma_{|[q,p]}) + \alpha_0.(p-q) \leq m(\gamma(q),\gamma(p)) \leq A_L(\gamma_{|[q,p]}) + \alpha_0.(p-q)$$

We deduce equality everywhere. And since the integers $p$ and $q$ are arbitrary, we conclude from the last point of Remark 5.2.18 that the curve $\gamma$ is semi-static and $(s,\gamma(s),\dot\gamma(s)) = (s,\tilde{x})$ belongs to the Mañé set $\tilde{\mathcal{N}}$. □

We devote the rest of this subsection to understanding the dynamical behaviour of the semi-static curves in the Mañé set $\tilde{\mathcal{N}}$. To achieve this, we need to introduce additional concepts following [Mn97, CDI97, CI99].

**Definition 5.2.22.** 1. We define the semi-distance $d_1 : \mathcal{M}_0 \times \mathcal{M}_0 \to \mathbb{R}_{\geq 0}$ as

$$d_1(x,y) = h^\infty(x,y) + h^\infty(y,x) \tag{5.2.37}$$

And set $\sim$ to be the equivalence relation on $\mathcal{M}_0$ given by

$$x \sim y \iff d_1(x,y) = 0 \tag{5.2.38}$$

The *static classes* are the equivalence classes of the equivalence relation $\sim$. We denote by $\mathbb{M}$ the set of static classes.

2. For a curve $\gamma : \mathbb{R} \to M$, we define its integer-time $\alpha$-limit set $\alpha_k(\gamma)$ and its its integer-time $\omega$-limit set $\omega_k(\gamma)$ by

$$\begin{aligned}\alpha_k(\gamma) &= \{z \in M \mid \exists (q_n)_n \in \mathbb{N}^{\mathbb{N}};\ \lim_n q_n = +\infty,\ \lim_n \gamma(-k.q_n) = z\}\\ \omega_k(\gamma) &= \{z \in M \mid \exists (q_n)_n \in \mathbb{N}^{\mathbb{N}};\ \lim_n q_n = +\infty,\ \lim_n \gamma(k.q_n) = z\}\end{aligned} \tag{5.2.39}$$

Then, we set a partial order $\preceq$ on $\mathbb{M}$ given by

i. $\preceq$ is reflexive and transitive.

ii. For all $x$ and $y$ in $\mathbb{M}$, if there exists a curve $\gamma$ with lift $(\gamma,\dot\gamma)$ in the Mañé set $\tilde{\mathcal{N}}$ such that its integer $\alpha$-limit set $\alpha_1(\gamma)\subset x$ and its integer $\omega$-limit set $\omega_1(\gamma)\subset y$, then $x\le y$.

**Remark 5.2.23.** Note that the semi-distance $d_1$ is non-negative thanks to the triangular inequality property (5.2.28) and the non-negativity Property 7 of Proposition 5.2.14. Indeed, for all $x$ and $y$ in $\mathcal{M}_0$,

$$d_1(x,y)=h^\infty(x,y)+h^\infty(y,x)\ge h^\infty(x,x)\ge 0$$

It is a distance when restricted to the set $\mathbb{M}$.

**Proposition 5.2.24.** *If $x$ and $y$ are two point of the Peierls set $\mathcal{A}_0$ with the same static class in $\mathbb{M}$, then for all point $z\in M$,*

$$h^\infty(x,z)=h^\infty(x,y)+h^\infty(y,z) \tag{5.2.40}$$

*Proof.* This is no more than an application of the triangular inequality (5.2.28) as follows

$$\begin{aligned}h^\infty(x,z)&\le h^\infty(x,y)+h^\infty(y,z)\\&\le h^\infty(x,y)+h^\infty(y,x)+h^\infty(x,z)\\&=d_1(x,y)+h^\infty(x,z)=h^\infty(x,z)\end{aligned}$$

So, there is equality everywhere. □

The following proposition motivates the definition of the partial order $\le$.

**Proposition 5.2.25.** *Let $\gamma: I\to M$ be a semi-static curve. Then, whenever the $\alpha$ and $\omega$-limit sets are well-defined for the interval $I$, there exist static classes $\bar x$ and $\bar y$ in $\mathbb{M}$ such that*

$$\alpha_1(\gamma)\subset\bar x\subset\mathcal{A}_0\quad\textit{and}\quad\omega_1(\gamma)\subset\bar y\subset\mathcal{A}_0 \tag{5.2.41}$$

*Proof.* we only prove the first inclusion. Assume that $I$ is unbounded in the negative direction. Let $z$ be an element of $\alpha_1(\gamma)$ and let $q_n$ be an increasing sequence of positive integers such that $\lim_n q_{n+1}-q_n=+\infty$ and $\gamma(-q_n)$ converges to $z$. We first prove that $z$ belongs to $\mathcal{A}_0$. Fix a small time $s>0$. By the A priori compactness Theorem 9.5.11, we know that the curves $\gamma_n(t)=\gamma(-q_n+t)$ are bounded in the $C^1$-topology. Thus, we can assume, up to extraction, that they $C^1$-converge on the interval $[0,s]$ to a curve $\sigma_z:[0,s]\to M$ with $\sigma_z(0)=z$. Let us evaluate $h^{\infty+t}(z,\sigma_z(t))$. The regularity Proposition 5.1.13 shows that for all integers $n\ge 0$

$$|h^{q_{n+1}-q_n+t}(z,\sigma_z(t))-h^{q_{n+1}-q_n+t}(\gamma(t-q_{n+1}),\gamma(-q_n))|\le\kappa_1.[d(z,\gamma(t-q_{n+1}))+d(\sigma_z(t),\gamma(-q_n))]$$

which implies that

$$\liminf_n h^{q_{n+1}-q_n+t}(z,\sigma_z(t)) = \liminf_n h^{q_{n+1}-q_n+t}(\gamma(-q_{n+1}),\gamma(t-q_n))$$

Using the semi-static behaviour of the curve $\gamma$, we get the following

$$\begin{aligned} h^{\infty+t}(z,\sigma_z(t)) &\leq \liminf_n h^{q_{n+1}-q_n+t}(z,\sigma_z(t)) = \liminf_n h^{q_{n+1}-q_n+t}(\gamma(-q_{n+1}),\gamma(t-q_n)) \\ &= \liminf_n A_L(\gamma_{|[-q_{n+1},t-q_n]}) = \liminf_n m^t(\gamma(-q_{n+1}),\gamma(t-q_n)) = m^t(z,\sigma_z(t)) \end{aligned}$$

where the last equality is due to the continuity of the Mañé potential $m^t$ stated in Proposition 5.2.19. And since $m^t(z,\sigma_z(t)) \leq h^{\infty+t}(z,\sigma_z(t))$, we deduce the equality $m^t(z,\sigma_z(t)) = h^{\infty+t}(z,\sigma_z(t))$ for all times $t \in (0,s]$. Taking the time $t$ to zero, we get

$$h^\infty(z,z) = \lim_{t\to 0} h^{\infty+t}(z,\sigma_z(t)) = \lim_{t\to 0} m^t(z,\sigma_z(t)) \leq \lim_{t\to 0} A_L(\sigma_{z|[0,t]}) = 0 \tag{5.2.42}$$

where the first limit is due to the uniform continuity of $h^\infty$ on the graph of $(t,\sigma_z(t))$ for $t \in [0,s]$. Hence, we deduce from the non-negativity Property 7 of Proposition 5.2.14 that $h^\infty(z,z) = 0$ meaning that $z$ belongs to the Peierls set $\mathcal{A}_0$.

Now we show that the $\alpha$-limit set $\alpha_1(\gamma)$ belongs to a unique static class. Let $x$ and $z$ be two points of $\alpha_1(\gamma)$. Let $q_n$ and $p_n$ be two intertwined increasing sequences of integers such that $\gamma(-q_n)$ converges to $x$, $\gamma(-p_n)$ converges to $z$, for all $n \geq 0$, $q_n < p_n < q_{n+1}$ and $\lim_n q_{n+1} - p_n = \lim_n p_n - q_n = +\infty$. As done above, we construct two curves $\sigma_z$ and $\sigma_x : [0,s] \to M$ and we prove that

$$h^{\infty+t}(x,\sigma_z(t)) = m^t(x,\sigma_z(t)) \quad \text{and} \quad h^{t,\infty+2t}(\sigma_z(t),\sigma_x(2t)) = m^{t,2t}(\sigma_z(t),\sigma_x(2t))$$

Thus, we get

$$\begin{aligned} h^{\infty+t}(x,\sigma_z(t)) + h^{t,\infty+2t}(\sigma_z(t),\sigma_x(2t)) &= m^t(x,\sigma_z(t)) + m^{t,2t}(\sigma_z(t),\sigma_x(2t)) \\ &= \lim_n m^t\big(\gamma(-q_{n+1}),\gamma(t-p_n)\big) + m^{t,2t}\big(\gamma(t-p_n),\gamma(2t-q_n)\big) \\ &= \lim_n A_{L+\alpha_0}(\gamma_{|[-q_{n+1},t-p_n]}) + \lim_n A_{L+\alpha_0}(\gamma_{|[t-p_n,2t-q_n]}) \\ &= \lim_n A_{L+\alpha_0}(\gamma_{|[-q_{n+1},2t-q_n]}) = \lim_n m^{2t}\big(\gamma(-q_{n+1}),\gamma(2t-q_n)\big) \\ &= m^{2t}(x,\sigma_x(2t)) \end{aligned}$$

And taking $t$ to 0, we finally get

$$\begin{aligned} d_1(x,z) = h^\infty(x,z) + h^\infty(z,x) &= \lim_{t\to 0} h^{\infty+t}(x,\sigma_z(t)) + h^{t,\infty+2t}(\sigma_z(t),\sigma_x(2t)) \\ &= \lim_{t\to 0} m^{2t}(x,\sigma_x(2t)) \leq \lim_{t\to 0} A_{L+\alpha_0}(\sigma_{x|[0,2t]}) = 0 \end{aligned}$$

However, we already know from Remark 5.2.23 that $d_1(x,z) \geq 0$. We obtain $d_1(x,z) = 0$ and the two points $x$ and $z$ belong to the same static class $\bar{x}$. □

**Proposition 5.2.26.** *Let $\gamma : \mathbb{R} \to M$ be a semi-static curve on $M$ and let $\bar{x}$ be the static class of $\alpha_1(\gamma)$. Then for all point $x$ of $\bar{x}$, the curve $\gamma$ is calibrated by the weak-KAM solution $h^\infty(x,\cdot)$.*

*Proof.* Fix two real times $s < t$. Let us first assume that $x$ belongs to $\alpha_1(\gamma)$. Then there exists an increasing sequence of integers $q_n$ such that $\gamma(-q_n)$ converges to $x$. We have from the definition of semi-static curves that

$$\begin{aligned} m^t(\gamma(-q_n),\gamma(t)) &= A_{L+\alpha_0}(\gamma_{|[-q_n,t]}) = A_{L+\alpha_0}(\gamma_{|[-q_n,s]}) + A_{L+\alpha_0}(\gamma_{|[s,t]}) \\ &= m^s(\gamma(-q_n),\gamma(s)) + A_{L+\alpha_0}(\gamma_{|[s,t]}) \end{aligned}$$

In order to take the limit, we need to make sure that we work in the domain of continuity of the Mañé potential $m^{s,t}$. It suffices to assume that $s$ and $t \neq 0$ in $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$. Then we can take the limit on $n$ to get

$$m^t(x,\gamma(t)) = m^s(x,\gamma(s)) + A_{L+\alpha_0}(\gamma_{|[s,t]})$$

We need to replace $m^t$ by $h^{\infty+t}$. To do so, remark that for all integer $n \geq 0$, we have from the triangular inequality (5.2.29) of the Peierls Barrier that

$$h^{\infty+t}(x,\gamma(t)) = h^{\infty+t+n}(x,\gamma(t)) \leq h^\infty(x,x) + h^{t+n}(x,\gamma(t)) = h^{t+n}(x,\gamma(t))$$

where we used the inclusion $x \in \mathcal{A}_0$ derived from Proposition 5.2.25. And taking the infimum over $n$, we get the inequality $h^{\infty+t}(x,\gamma(t)) \leq m^t(x,\gamma(t))$. And since the inverse inequality is immediate, we deduce the equality $h^{\infty+t}(x,\gamma(t)) = m^t(x,\gamma(t))$. Doing the same for $s$, we finally get

$$h^{\infty+t}(x,\gamma(t)) = h^{\infty+s}(x,\gamma(s)) + A_{L+\alpha_0}(\gamma_{|[s,t]})$$

Now if for example $s = 0$ in $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ and $\gamma(s) = x$, we take $s' > s$ and we get $h^{\infty+s'}(x,\gamma(t)) = m^{s'}(x,\gamma(s'))$. The regularity of the Peierls barrier $h^\infty$ allows to take the limit $s' \to s$ and conclude.

The extension of the result to the static class $\bar{x}$ is an immediate application of Proposition 5.2.24. □

**Remark 5.2.27.** Gathering Propositions 5.2.20 and 5.2.26, we infer the following characterization of semi-static curves : a curve is semi-static if and only if it is calibrated by a weak-KAM solution.

The following theorem is an adaptation in the non-autonomous case of a result proved in Section 3.11 of [CI99]. It can also be derived from [Ber08] which covers the time-dependant framework.

**Theorem 5.2.28.** *If the set of static classes* $\mathbb{M}$ *is finite, then for all* $x$ *and* $y$ *in* $\mathbb{M}$ *we have* $x \preceq y$.

**Remark 5.2.29.** The general result true even in the case where $\mathbb{M}$ is infinite is the chain-transitivity of the Mañé set $\tilde{\mathcal{N}}$. However, in Section 8.3, we will need the specific behaviour of the semi-static curves between the static classes as stated by the theorem.

# Chapitre 6

# Action of the Lax-Oleinik Operator $\mathcal{T}$ on its Non-Wandering Set $\Omega(\mathcal{T})$

After introducing the preliminary tools from non-autonomous weak-KAM theory, this chapter is dedicated to studying the behavior of the Lax-Oleinik operator $\mathcal{T}$, focusing specifically on the properties of its restriction to the non-wandering set $\Omega(\mathcal{T})$.

In Proposition 5.1.13, we examined a regularizing property of the semi-group $\mathbb{T}$, which ensures the equi-Lipschitz regularity of viscosity solutions. In this chapter, however, we establish only the non-expansiveness of the semi-group and analyze the implications of this result for the asymptotic behavior of $\mathcal{T}$.

A key outcome is that for viscosity solutions, being non-wandering is equivalent to being recurrent (see Proposition 6.3.3). Additionally, we show that $\mathcal{T}$ acts as an isometry on the non-wandering set $\Omega(\mathcal{T})$, as demonstrated by the following proposition.

**Proposition 6.0.1.**

1. *The restriction of $\mathcal{T}$ to the non-wandering set $\Omega(\mathcal{T})$ is an isometric bijection, i.e $\mathcal{T}$ is invertible and for all $v$ and $w$ in $\Omega(\mathcal{T})$, $\|\mathcal{T}v - \mathcal{T}w\|_\infty = \|v - w\|_\infty$.*
2. *More generally, if we denote by $\Omega_\tau(\mathcal{T}) = \mathcal{T}^\tau\big(\Omega(\mathcal{T})\big)$, then for all times $s < t$, the operator $\mathcal{T}^{s,t} : \Omega_s(\mathcal{T}) \to \Omega_t(\mathcal{T})$ is an isometric bijection. We denote its inverse map by $\mathcal{T}^{t,s}$.*
3. *For all times $s$, $t$ and $\tau$ in $\mathbb{R}$, $\mathcal{T}^{s,t} = \mathcal{T}^{\tau,t} \circ \mathcal{T}^{s,\tau}$.*

This property, demonstrated in Section 6.3, arises from the fact that the Lax-Oleinik operator $\mathcal{T}$ is both surjective and non-expanding on the $\omega$-limit sets of scalar maps, which are compact. It follows from a classical result that any surjective, non-expanding map on a compact set is a bijective isometry. It is a classical result that surjective, non-expanding maps on compact sets are bijective isometries.

From this proposition, we establish that all non-wandering elements of $\Omega(\mathcal{T})$ can be associated with global viscosity solutions of the form $u(t,x) = \mathcal{T}^{0,t}u(x) : \mathbb{R}\times M \to \mathbb{R}$, where $u(0,\cdot) = u$. If we denote by $\mathcal{B}(\mathcal{T})$ the set of bounded global viscosity solutions defined for all times $t \in \mathbb{R}$, then we demonstrate the following characterization of non-wandering viscosity solutions.

**Theorem 6.0.2.** *A global viscosity solution of the Hamilton-Jacobi equation 5.0.1 is bounded if and only if it is non-wandering. In other words, the following equality holds*

$$\Omega(\mathcal{T}) = \mathcal{B}(\mathcal{T}) \tag{6.0.1}$$

A natural question that arises is that of...

**Question 6.0.3.** Giving an explicit example of a global viscosity solution that is unbounded in negative times.

## 6.1 Non-Expansiveness and Boundedness of $\mathcal{T}$

We begin with a proof of the well-known non-expansiveness of the Lax-Oleinik operator $\mathcal{T}$.

**Proposition 6.1.1.** *For all time $t > 0$, The time $t$ Lax-Oleinik operators $\mathcal{T}_0^t$ and $\mathcal{T}^t$ are non-expanding, i.e*

$$\forall u, v \in \mathcal{C}(M,\mathbb{R}), \quad \|\mathcal{T}^t u - \mathcal{T}^t v\|_\infty = \|\mathcal{T}_0^t u - \mathcal{T}_0^t v\|_\infty \leq \|u - v\|_\infty \tag{6.1.1}$$

*In particular, the Lax-Oleinik operators $\mathcal{T}_0$ and $\mathcal{T}$ are non-expanding.*

*Proof.* First, note that

$$\mathcal{T}^t u - \mathcal{T}^t v = \mathcal{T}_0^t u + \alpha_0.t - \mathcal{T}_0^t v - \alpha_0.t = \mathcal{T}_0^t u - \mathcal{T}_0^t v$$

We prove the non-expansiveness of $\mathcal{T}_0^t$. Let $x \in M$. Corollary 5.1.10 provides with a minimizing curve $\sigma : [0,t] \to M$ with $\sigma(t) = x$ realizing the infimum in the definition (5.1.5) of

$\mathcal{T}_0^t v(x)$. Hence we get

$$\begin{aligned}\mathcal{T}_0^t u(x) - \mathcal{T}_0^t v(x) &= \inf_{\substack{\gamma : [0,t] \to M \\ t \mapsto x}} \left\{ u(\gamma(0)) + \int_0^t L(s,\gamma(s),\dot\gamma(s))\, ds \right\} - \left( v(\sigma(0)) + \int_0^t L(s,\sigma(s),\dot\sigma(s))\, ds \right) \\ &\le \left( u(\sigma(0)) + \int_0^t L(s,\sigma(s),\dot\sigma(s))\, ds \right) - \left( v(\sigma(0)) + \int_0^t L(s,\sigma(s),\dot\sigma(s))\, ds \right) \\ &\le u(\sigma(0)) - v(\sigma(0)) \le \|u - v\|_\infty\end{aligned}$$

Since $u$ and $v$ play symmetric roles, we get the lacking inequality. □

As a direct consequence, we obtain the forward-time boundedness of all viscosity solutions.

**Corollary 6.1.2.** *For all $(s,u) \in \mathbb{R} \times \mathcal{C}(M,\mathbb{R})$, the family $\left(u(t,x) = \mathcal{T}^{s,t} u\right)_{t \ge s}$ is uniformly bounded in $\mathcal{C}(M,\mathbb{R})$.*

*Proof.* Fix an element $(s,u) \in \mathbb{R} \times \mathcal{C}(M,\mathbb{R})$. Let $v$ be a weak-KAM solution which existence will be proved in Corollary 6.2.4. By the non-expansiveness Proposition 6.1.1, we get for all $t \ge s$

$$||\mathcal{T}^{s,t} u - \mathcal{T}^{s,t} \mathcal{T}^s v||_\infty \le ||u - \mathcal{T}^s v||_\infty$$

And we know from the definition of weak-KAM solutions that

$$\mathcal{T}^{s,t} \mathcal{T}^s v = \mathcal{T}^t v = \mathcal{T}^{\lfloor t \rfloor, t} \mathcal{T}^{\lfloor t \rfloor} v = \mathcal{T}^{\lfloor t \rfloor, t} v = \mathcal{T}^{t - \lfloor t \rfloor} v$$

Hence, we obtain from the continuity in time of $\mathcal{T}^\tau v$ that

$$\begin{aligned}||\mathcal{T}^{s,t} u||_\infty &\le ||u - \mathcal{T}^s v||_\infty + ||\mathcal{T}^{s,t}\mathcal{T}^s v||_\infty = ||u - \mathcal{T}^s v||_\infty + ||\mathcal{T}^{t-\lfloor t \rfloor} v||_\infty \\ &\le ||u - \mathcal{T}^s v||_\infty + \sup_{\tau \in [0,1]} ||\mathcal{T}^\tau v||_\infty < +\infty\end{aligned}$$

□

**Remark 6.1.3.** This boundedness result offers a characterization of the Mañé critical value $\alpha_0$. Indeed, $\alpha_0$ is the unique real constant $c$ such that there exists (or for all) scalar map $u \in \mathcal{C}(M,\mathbb{R})$, the family $\left(u_c(t,x) = \mathcal{T}_0^{s,t} u + c.(t-s)\right)_{t \ge s}$ is uniformly bounded.

## 6.2 Stability of Viscosity Solutions

We present stability results for viscosity solutions of a Tonelli Hamiltonian. The stability of limits given by Proposition 6.2.2 remains valid in more general frameworks. For a more detailed exposition on stability results for viscosity solutions, see [Bar94]. However, in the Tonelli framework, the Lax-Oleinik operator $\mathcal{T}$ allows for the stability of both the infimum and the liminf.

**Proposition 6.2.1.** *Let $(v_i : [s,t] \times M \to \mathbb{R})_{i \in I}$ be a family of viscosity solutions and let $u(t,x) = \inf_{i \in I}\{v_i(t,x)\}$. Then $u$ is a viscosity solution.*

*Proof.* Fix two real times $s \leq s' < t' \leq t$. Then, for all $x \in M$ the following computation holds

$$\begin{aligned}
\mathcal{T}^{s',t'}u(s',x) &= \inf_{y \in M}\left\{u(s',y) + h^{s',t'}(y,x)\right\} \\
&= \inf_{y \in M}\left\{\inf_{i \in I}\{v_n(s',y)\} + h^{s',t'}(y,x)\right\} \\
&= \inf_{i \in I}\ \inf_{y \in M}\left\{v_n(s',y) + h^{s',t'}(y,x)\right\} \\
&= \inf_{i \in I}\left\{\mathcal{T}^{s',t'}v_n(s',x)\right\} = \inf_{n \geq 0}\{v_n(t',x)\} = u(t',x)
\end{aligned}$$

□

**Proposition 6.2.2.** *Let $v_n$ be a sequence of scalar maps in $\mathcal{C}(M,\mathbb{R})$ that converges to $v$. Then, for all times $s \in \mathbb{R}$, the associated viscosity solutions $v_n : [s,+\infty) \times M \to \mathbb{R}$ defined by $v_n(t,x) = \mathcal{T}^{s,t}v_n(x)$ converge to the viscosity solution $v(t,x) = \mathcal{T}^{s,t}v(x)$ in the $C^0$ topology.*

*Proof.* For all times $s < t$, we have by non-expansiveness of $\mathcal{T}^{s,t}$ that

$$\|v_n(t,\cdot) - v(t,\cdot)\|_\infty = \|\mathcal{T}^{s,t}v_n - \mathcal{T}^{s,t}v\|_\infty \leq \|v_n - v\|_\infty \longrightarrow 0 \quad \text{as } n \to +\infty \tag{6.2.1}$$

□

**Proposition 6.2.3.** *Let $u \in \mathcal{C}(M,\mathbb{R})$ be a scalar map with associated viscosity solution $u(t,x) : [0,+\infty) \times M \to \mathbb{R}$ and let $(k_n)_n$ be an increasing sequence of integers. We define the map $v : \mathbb{R} \times M \to \overline{\mathbb{R}}$ defined by*

$$v(t,x) = \liminf_n u(t + k_n, x) \tag{6.2.2}$$

*If $v$ is everywhere finite, then it is a viscosity solution of the Hamilton-Jacobi equation.*

*Proof.* We start by proving that it is a sub-solution i.e. for all times $s \leq t$, $\mathcal{T}^{s,t}v(s,\cdot) \leq v(t,\cdot)$. Fix $(t,x)$ in $\mathbb{R} \times M$ and subsequence $k_{n_i}$ of $k_n$ such that $\mathcal{T}^{t+k_{n_i}}u(x)$ converges to $v(t,x)$. For every integer $i$, we get from Tonelli's Theorem 5.1.9 a minimizing curve $\gamma_i : [s,t] \to M$ which realizes $\mathcal{T}^{s,t}(\mathcal{T}^{s+k_{n_i}}u)(x)$ i.e. such that $\gamma_i(t) = x$ and

$$\mathcal{T}^{t+k_{n_i}}u(x) = \mathcal{T}^{s,t}(\mathcal{T}^{s+k_{n_i}}u)(x) = \mathcal{T}^{s+k_{n_i}}u(\gamma_i(s)) + \int_s^t \Big(L(\tau,\gamma_i(\tau),\dot{\gamma}_i(\tau)) + \alpha_0\Big)\, d\tau \tag{6.2.3}$$

These curves $\gamma_i$ are minimizing. Thus, by Corollary 5.1.12, we can assume up to extraction that the curves $\gamma_i$ converge to $\gamma : [s,t] \to M$ in the $C^1$-topology with $\gamma(t) = \gamma_i(t) = x$. Therefore, we get

$$\begin{aligned} v(t,x) = \lim_i \mathcal{T}^{t+k_{n_i}} u(x) &\geq \liminf_i \mathcal{T}^{s+k_{n_i}} u(\gamma_i(s)) + \lim_i \int_s^t \Big( L(\tau, \gamma_i(\tau), \dot\gamma_i(\tau)) + \alpha_0 \Big)\, d\tau \\ &= \liminf_i \mathcal{T}^{s+k_{n_i}} u(\gamma_i(s)) + \int_s^t \Big( L(\tau, \gamma(\tau), \dot\gamma(\tau)) + \alpha_0 \Big)\, d\tau \end{aligned}$$

However, we know from Corollary 5.1.14 that the family $(\mathcal{T}^t u)_t$ is equicontinuous, which yields

$$|\mathcal{T}^{s+k_{n_i}} u(\gamma_i(s)) - \mathcal{T}^{s+k_{n_i}} u(\gamma(s))| \leq \kappa_1 . |u(\gamma_i(s)) - u(\gamma(s))| \longrightarrow 0 \quad \text{as } n \to \infty$$

Hence,

$$\liminf_i \mathcal{T}^{s+k_{n_i}} u(\gamma_i(s)) = \liminf_i \mathcal{T}^{s+k_{n_i}} u(\gamma(s)) \geq \liminf_n \mathcal{T}^{s+k_n} u(\gamma(s)) = v(s, \gamma(s))$$

and

$$v(t,x) \geq v(s,\gamma(s)) + \int_s^t \Big( L(\tau, \gamma(\tau), \dot\gamma(\tau)) + \alpha_0 \Big)\, d\tau \geq \mathcal{T}^{s,t} v(s,x)$$

We now establish the inverse inequality. Let $\gamma : [s,t] \to M$ be any curve with $\gamma(t) = x$. We know from the definition of the Lax-Oleinik operator that

$$\mathcal{T}^{t+k_n} u(x) \leq \mathcal{T}^{s+k_n} u(\gamma(s)) + \int_s^t \Big( L(\tau, \gamma(\tau), \dot\gamma(\tau)) + \alpha_0 \Big)\, d\tau$$

and taking the liminf, we get that for each such curve $\gamma$,

$$v(t,x) \leq v(s,\gamma(s)) + \int_s^t \Big( L(\tau, \gamma(\tau), \dot\gamma(\tau)) + \alpha_0 \Big)\, d\tau$$

which gives the desired inequality

$$v(t,x) \leq \mathcal{T}^{s,t} v(s,x)$$

□

**Corollary 6.2.4.** *There exists a weak-KAM solution of the Hamilton-Jacobi equation* (5.0.1).

*Proof.* Fix a point $x_0$ in $M$ and set $u(t,x) = h^t(x_0,x)$. We know from Proposition 5.2.7 that $u : (0,+\infty) \times M \to \mathbb{R}$ is a viscosity solution of the Hamilton-Jacobi equation (5.0.1). Moreover, we know due to Proposition 5.2.10 that the map $v(t,x) = \liminf_n u(t+n,x) = h^{\infty+t}(x_0,x)$ is everywhere finite. Hence, we infer from Proposition 6.2.3 that map $v(t,x) =$

$\liminf_n u(t+n,x) = h^{\infty+t}(x_0,x)$ is also a viscosity solution of (5.0.1). Furthermore, we have

$$\mathcal{T}v = v(t+1,\cdot) = \liminf_n u(t+1+n,\cdot) = \liminf_n u(t,\cdot) = v \tag{6.2.4}$$

Hence, $v$ belongs to $\mathrm{Fix}(\mathcal{T})$ and it is a weak-KAM solution. □

**Remark 6.2.5.** Note that in the statement of Proposition 6.2.3 we didn't assume that $v$ is finite as this is a consequence of Proposition 6.1.2 which is itself a consequence of Corollary 6.2.4.

Note also that the solution $v$ constructed in the proof of this corollary is the weak-KAM solution given by the Peierls barrier $h^\infty(x_0,\cdot)$ introduced in Section 5.2.3.

## 6.3 Restriction to the Non-Wandering Set $\Omega(\mathcal{T})$

We explore the implications of the non-expansiveness of the Lax-Oleinik operator $\mathcal{T}$ on its non-wandering and recurrent sets. We start by introducing the definitions of some asymptotic objects of $\mathcal{T}$.

To a scalar map $u$ of $\mathcal{C}(M,\mathbb{R})$, we associate its *$\omega$-limit set* $\omega(u) \coloneqq \omega_{\mathcal{T}}(u)$ under the operator $\mathcal{T}$ as the set of limit points in the $\mathcal{C}(M,\mathbb{R})$ of the sequence $(\mathcal{T}^n(u))_{n\in\mathbb{N}}$. More precisely

$$\omega(u) = \{v \in \mathcal{C}(M,\mathbb{R}) \mid \exists (k_n)_n \in \mathbb{N}^{\mathbb{N}} \text{ increasing sequence s.t } \|\mathcal{T}^{k_n}u - v\|_\infty \to 0 \text{ as } n \to \infty\} \tag{6.3.1}$$

**Definition 6.3.1.**
1. The *recurrent set* $\mathcal{R}(\mathcal{T})$ of $\mathcal{T}$ is the set of $\mathcal{T}$-recurrent elements of $\mathcal{C}(M,\mathbb{R})$ i.e. the set of $u \in \mathcal{C}(M,\mathbb{R})$ such that $u$ belongs to $\omega(u)$.
2. Let $\underline{p} = (p_n)_n$ be an increasing of $\mathbb{N}$. An element $u$ of $\mathcal{R}(\mathcal{T})$ is said *$\underline{p}$-recurrent* if $\lim_n \mathcal{T}^{p_n}u = u$.
3. The *non-wandering set* $\Omega(\mathcal{T})$ of $\mathcal{T}$ is the set of elements $u$ of $\mathcal{C}(M,\mathbb{R})$ such that for every neighbourhood $U$ of $u$ in $\mathcal{C}(M,\mathbb{R})$, there are infinitely many positive integers $k$ such that $\mathcal{T}^k(U)\cap U \neq \varnothing$.

**Remark 6.3.2.** Every element $u$ of $\mathcal{R}(\mathcal{T})$ or $\Omega(\mathcal{T})$ corresponds to a recurrent or a non-wandering (for integer times) viscosity solution $u(t,x)$. By abuse of language, we will often refer to the sets $\mathcal{R}(\mathcal{T})$ and $\Omega(\mathcal{T})$ as the sets of recurrent and non-wandering viscosity solutions, respectively.

As a consequence of the non-expansiveness of $\mathcal{T}$ shown in Proposition 6.1.1, we get the following properties

**Proposition 6.3.3.** *1. Let $u \in \mathcal{C}(M,\mathbb{R})$. Then, its $\omega$-limit set $\omega(u)$ is compact in $\mathcal{C}(M,\mathbb{R})$, and the restriction of $\mathcal{T}$ to $\omega(u)$ is minimal, i.e. for all $v \in \omega(u)$, $\omega(u) = \overline{\{\mathcal{T}^n v \mid n \in \mathbb{N}\}} = \omega(v)$.*

*2. The non-wandering set $\Omega(\mathcal{T})$ is equal to the recurrent set $\mathcal{R}(\mathcal{T})$.*

*3. The relation $u \sim v \Leftrightarrow v \in \omega(u)$ is an equivalence relation. If we denote by $\Lambda$ the set of its equivalence classes, then we have*

$$\Omega(\mathcal{T}) = \mathcal{R}(\mathcal{T}) = \bigsqcup_{u\in\Lambda} \omega(u) \tag{6.3.2}$$

*where the union is disjoint.*

*Proof.* 1. The set $\omega(u)$ is closed and $\mathcal{T}$-invariant. Let us show that it is bounded in $\mathcal{C}(M,\mathbb{R})$. For all positive integer $n \geq 0$ and for all $x$ in $M$, if we denote by $y$ the point that realizes the infimum in $\mathcal{T}^n u(x)$, then we have

$$|\mathcal{T}^n u(x)| = |u(y) + h^n(y,x)| \leq \|v\|_\infty + \max(\|m\|_\infty, \|M\|_\infty) \leq \|u\|_\infty + \kappa_1.2\,\mathrm{diam}(M)$$

where we used the bound on $m = \inf_n h^n$ and $M = \sup_n h^n$ found in Proposition 5.2.10. Taking limits on $n$, we observe that this bound holds for every element of $\omega(u)$. Moreover, since we know from Corollary 5.1.14 that the maps $(u(t,\cdot))_{t\geq 1}$ are equilipschitz, we infer that all the elements of $\omega(u)$ are $\kappa_1$-lipschitz. Therefore, the Arzéla-Ascoli theorem asserts that this set is compact in $\mathcal{C}(M,\mathbb{R})$.

We show the minimality property. Let $v \in \omega(u)$ and $(n_k)_{k\in\mathbb{N}}$ be an increasing sequence of integers such that $\mathcal{T}^{n_k}u$ converges to $v \in \omega(u)$. The set $\omega(u)$ is closed and $\mathcal{T}$-invariant. Hence, we have $\overline{\{\mathcal{T}^n v \mid n \in \mathbb{N}\}} \subset \omega(u)$.

To prove the inverse inclusion, fix $w \in \omega(u)$. There exists an increasing sequence $(p_k)_{k\in\mathbb{N}}$ such that $\mathcal{T}^{n_k+p_k}u \to w$. The non-expansiveness (6.1.1) of the semi-group $\mathcal{T}$ results in

$$\begin{aligned}\|\mathcal{T}^{p_k}v - w\|_\infty &\leq \|\mathcal{T}^{p_k}v - \mathcal{T}^{n_k+p_k}u\|_\infty + \|\mathcal{T}^{n_k+p_k}u - w\|_\infty \\ &\leq \|v - \mathcal{T}^{n_k}u\|_\infty + \|\mathcal{T}^{n_k+p_k}u - w\|_\infty \longrightarrow 0 \quad \text{as } k \to \infty\end{aligned}$$

Hence, $w \in \overline{\{\mathcal{T}^n v \mid n \in \mathbb{N}\}}$.

By closedness and $\mathcal{T}$-invariance, we know that $\omega(v) \subset \omega(u)$. Hence, the minimality gives the inverse inclusion and the equality of sets.

2. The inclusion $\mathcal{R}(\mathcal{T}) \subset \Omega(\mathcal{T})$ is immediate. We need to prove the inverse inclusion. Let $u \in \Omega(\mathcal{T})$ be a non-wandering element under $\mathcal{T}$. We aim to prove that it is recurrent. For all positive integer $n > 0$, consider the set $U_n = \left\{v \in \mathcal{C}(M,\mathbb{R}) \mid \|v-u\|_\infty < \frac{1}{n}\right\}$. Using the non-wandering property of $u$, we inductively construct an increasing sequence of positive

integers $k_n$ such that $\mathcal{T}^{k_n} U_n \cap U_n \neq \varnothing$. Let $v_n$ be an element of $U_n \cap \mathcal{T}^{-k_n} U_n \neq \varnothing$. The properties on $U_n$ translate on $v_n$ as follows

$$\|v_n - u\|_\infty < \frac{1}{n} \quad \text{and} \quad \|\mathcal{T}^{k_n} v_n - u\|_\infty < \frac{1}{n} \tag{6.3.3}$$

Thus

$$\begin{aligned} \|\mathcal{T}^{k_n} u - u\|_\infty &\leq \|\mathcal{T}^{k_n} u - \mathcal{T}^{k_n} v_n\|_\infty + \|\mathcal{T}^{k_n} v_n - u\|_\infty \\ &\leq \|u - v_n\|_\infty + \|\mathcal{T}^{k_n} v_n - u\|_\infty < \frac{2}{n} \longrightarrow 0 \quad \text{as } n \to \infty \end{aligned}$$

where we used the non-expansiveness of $\mathcal{T}$ and $\mathcal{T}^{k_n}$ in the second line. We deduce that the sequence $(\mathcal{T}^{k_n} u)_n$ converges uniformly to $u$, meaning that $u$ belongs to $\omega(u)$. This is the definition of recurrence.

3. The fact that $\sim$ is an equivalence relation is due to the minimality of $\mathcal{T}$ on $\omega(u)$. And the equalities follow immediately from the two first properties. □

**Remark 6.3.4.**
1. It follows from (6.3.2) that the limit points of any viscosity solutions of (5.0.1) belong to a non-wandering element of $\Omega(\mathcal{T})$.
2. As an implication of the minimality on $\omega(u)$, we have that if $\omega(u)$ contains a periodic orbit, then it is equal to the orbit itself.

On every $\omega$-limit set $\omega(u)$, the Lax-Oleinik operator $\mathcal{T}$ is non-expanding and surjective in a compact set. This implies that $\mathcal{T}$ is a bijective isometry on $\omega(u)$. The Proposition 6.0.1 generalizes this result to the entire non-wandering set $\Omega(\mathcal{T})$ and allows for the definition of its inverse operator within it. We obtain a group $(\mathcal{T}^n)_{n\in\mathbb{Z}}$ of isometries acting on $\Omega(\mathcal{T})$.

*Proof of Proposition 6.0.1.* Let $v$ be an element of $\Omega(\mathcal{T})$. By recurrence, we know that there exists an increasing sequence of integers such that $\mathcal{T}^{k_n}(v)$ converges to $v$. Let $v'$ be a limit point in $\omega(v)$ of the sequence $\mathcal{T}^{k_n - 1}(v)$. This exists due to the compactness of $\omega(v)$ stated in the Property 1 of Proposition 6.3.3. Then, by continuity of the Lax-Oleinik operator $\mathcal{T}$, we get $\mathcal{T} v' = v$. We showed that the restriction of $\mathcal{T}$ to $\Omega(\mathcal{T})$ is onto.

Let us show that it is a bijective isometry on this set. Let $v$ and $w$ be two elements of $\Omega(\mathcal{T})$ and consider a sequences $(v_n, w_n)$ defined inductively as $(v_0, w_0) = (\mathcal{T} v, \mathcal{T} w)$, $(v_1, w_1) = (v, w)$ and for all integer $n \geq 1$, $(v_{n+1}, w_{n+1}) \in \mathcal{T}^{-1}(v_n, w_n)$. Let $k_n$ be an increasing sequence of integers such that $\lim_n k_{n+1} - k_n = +\infty$ and $(v_{k_n}, w_{k_n})$ converge to a point of the compact set $\omega(v) \times \omega(w)$. Then, we have

$$\|v_{k_{n+1} - k_n} - v_0\|_\infty = \|\mathcal{T}^{k_n} v_{k_{n+1}} - \mathcal{T}^{k_n} v_{k_n}\|_\infty \leq \|v_{k_{n+1}} - v_{k_n}\|_\infty \longrightarrow 0 \quad \text{as } n \to +\infty$$

giving the limit $\lim_n v_{k_{n+1}-k_n} = v_0 = \mathcal{T}v$. By symmetry, we also have $\lim_n w_{k_{n+1}-k_n} = w_0 = \mathcal{T}w$. Therefore, we obtain

$$\begin{aligned}
\|\mathcal{T}v - \mathcal{T}w\|_\infty \le \|v-w\|_\infty &= \|v_1 - w_1\|_\infty \\
&= \|\mathcal{T}^{k_{n+1}-k_n-1}v_{k_{n+1}-k_n} - \mathcal{T}^{k_{n+1}-k_n-1}w_{k_{n+1}-k_n}\|_\infty \\
&\le \|v_{k_{n+1}-k_n} - w_{k_{n+1}-k_n}\|_\infty \longrightarrow \|\mathcal{T}v - \mathcal{T}w\|_\infty \quad \text{as } n \to +\infty
\end{aligned}$$

The double inequality yields the isometric identity

$$\|\mathcal{T}v - \mathcal{T}w\|_\infty = \|v-w\|_\infty$$

The injectivity, and hence the bijectivity of $\mathcal{T}$ follow immediately.

For the general case, it suffices to see that for all times $s < t$, we have $\mathcal{T}^{\lfloor s\rfloor,\lceil t\rceil} = \mathcal{T}^{t,\lceil t\rceil} \circ \mathcal{T}^{s,t} \circ \mathcal{T}^{\lfloor s\rfloor,s}$ where $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ respectively stand for the floor and the ceil maps. Thus, for any $v$ and $w$ in $\Omega(\mathcal{T})$, we get

$$\begin{aligned}
\|v-w\|_\infty &= \|\mathcal{T}^{\lfloor s\rfloor,\lceil t\rceil}v - \mathcal{T}^{\lfloor s\rfloor,\lceil t\rceil}w\|_\infty \\
&= \|\mathcal{T}^{t,\lceil t\rceil} \circ \mathcal{T}^{s,t} \circ \mathcal{T}^{\lfloor s\rfloor,s}v - \mathcal{T}^{t,\lceil t\rceil} \circ \mathcal{T}^{s,t} \circ \mathcal{T}^{\lfloor s\rfloor,s}w\|_\infty \\
&\le \|\mathcal{T}^{s,t} \circ \mathcal{T}^{\lfloor s\rfloor,s}v - \mathcal{T}^{s,t} \circ \mathcal{T}^{\lfloor s\rfloor,s}w\|_\infty \\
&\le \|\mathcal{T}^{\lfloor s\rfloor,s}v - \mathcal{T}^{\lfloor s\rfloor,s}w\|_\infty \le \|v-w\|_\infty
\end{aligned}$$

We deduce equality everywhere, and the general result follows.

If $s < \tau < t$, then we have immediately that $\mathcal{T}^{s,t} = \mathcal{T}^{\tau,t} \circ \mathcal{T}^{s,\tau}$. If for example, $s < t < \tau$, then $\mathcal{T}^{s,\tau} = \mathcal{T}^{t,\tau} \circ \mathcal{T}^{s,t}$ and $\mathcal{T}^{s,t} = (\mathcal{T}^{t,\tau})^{-1} \circ \mathcal{T}^{s,\tau} = \mathcal{T}^{\tau,t} \circ \mathcal{T}^{s,\tau}$. The other cases are symmetric. □

## 6.4 Bounded Global Viscosity Solutions

This subsection is dedicated to proving Theorem 6.0.2, which characterizes the non-wandering elements of $\Omega(\mathcal{T})$ as corresponding to bounded global viscosity solutions.

**Definition 6.4.1.** 1. A *global viscosity solution* is a viscosity solution $u(t,x)$ defined for all times $t \in \mathbb{R}$.

2. We say that a scalar map $u \in \mathcal{C}(M,\mathbb{R})$ is *global* if it can be associated to a global viscosity solution $u(t,x)$ with $u(0,\cdot) = u$.

3. We define the set $\mathcal{B}(\mathcal{T})$ of global maps $u$ in $\mathcal{C}(M,\mathbb{R})$ which can be associated to a bounded global viscosity solution $u : \mathbb{R} \times M \to \mathbb{R}$ with $u(0,\cdot) = u$.

**Proposition 6.4.2.** *Every non-wandering element $v \in \Omega(\mathcal{T})$ is global and corresponds to a unique global viscosity solution $v(t,x)$ defined by*

$$v(t,x) = \mathcal{T}^t v(x) = \lim_n \mathcal{T}^{t+k_n} v(x) \tag{6.4.1}$$

*where $k_n$ is a sequence such that $v = \lim_n v(k_n,\cdot)$ and where the limit is uniform on $t$.*

*Proof.* Let us show uniqueness. If $v_1$ and $v_2$ are two global viscosity solutions with initial date $v_1(0,\cdot) = v_2(0,\cdot) = v \in \Omega(\mathcal{T})$. Then for all positive times $t > 0$, we have $v_1(t,\cdot) = \mathcal{T}^t v = v_2(0,\cdot)$ and

$$\mathcal{T}^{-t,0} v_1(-t,\cdot) = v_1(0,\cdot) = v = v_2(0,\cdot) = \mathcal{T}^{-t,0} v_2(-t,\cdot)$$

So that, by applying $\mathcal{T}^t = (\mathcal{T}^{-t,0})^{-1}$, we obtain $v_1 = v_2$.

Now set $v_1(t,x) = \mathcal{T}^t v(x)$. We have for all $s < t$, and by Proposition 6.0.1,

$$\mathcal{T}^{s,t} v_1(s,x) = \mathcal{T}^{s,t} \circ \mathcal{T}^s v(x) = \mathcal{T}^t v(x) = v_1(t,x)$$

Hence, $v_1$ is a global viscosity solution of (5.0.1). Moreover, since $\mathcal{T}^t$ is isometric on $\Omega(\mathcal{T})$, we get the limit

$$\|\mathcal{T}^{t+k_n} v - \mathcal{T}^t v\|_\infty = \|\mathcal{T}^{k_n} v - v\|_\infty \longrightarrow 0 \quad \text{as } n \to +\infty$$

yielding the second equality with uniform convergence in time. □

*Proof of Theorem 6.0.2.* We show double inclusion. We start with $\Omega(\mathcal{T}) \subset \mathcal{B}(\mathcal{T})$. Let $v$ be in $\Omega(\mathcal{T})$. By Proposition 6.3.3, we infer that for any integer $n \in \mathbb{Z}$, $v(n,\cdot)$ belongs to the compact set $\omega(v)$. Hence, for all real time $t$, we have

$$v(t,\cdot) = \mathcal{T}^{t-\lfloor t \rfloor} v(\lfloor t \rfloor,\cdot) \in \mathcal{T}^{t-\lfloor t \rfloor}(\omega(v)) \subset \mathcal{T}^{[0,1]}(\omega(v))$$

where the last set is compact in $\mathcal{C}(M,\mathbb{R})$ due to time continuity of the Lax-Oleinik operator $\mathcal{T}^t$. We deduce that $v$ belongs to $\mathcal{B}(\mathcal{T})$.

Now let $v$ be in $\mathcal{B}(\mathcal{T})$. The familiy $(v(-n,\cdot))_{n \geq 0}$ is bounded by definition of $\mathcal{B}(\mathcal{T})$ and equicontinuous by Corollary 5.1.14. Hence, applying the Arzelà-Ascoli theorem, there exists an increasing sequence of integers $k_n$ such that $\lim_n k_{n+1} - k_n = +\infty$ and $v(-k_n,\cdot)$ converges to a scalar map $v_\alpha$ in $\mathcal{C}(M,\mathbb{R})$. Set $k'_n = k_{n+1} - k_n$. We have

$$\begin{aligned}
\|\mathcal{T}^{k'_n} v_\alpha - v_\alpha\|_\infty &\leq \|\mathcal{T}^{k'_n} v_\alpha - v(-k_n,\cdot)\|_\infty + \|v(-k_n,\cdot) - v_\alpha\|_\infty \\
&\leq \|\mathcal{T}^{k'_n} v_\alpha - \mathcal{T}^{k'_n} v(-k_{n+1},\cdot)\|_\infty + \|v(-k_n,\cdot) - v_\alpha\|_\infty \\
&\leq \|v_\alpha - v(-k_{n+1},\cdot)\|_\infty + \|v(-k_n,\cdot) - v_\alpha\|_\infty \longrightarrow 0 \quad \text{as } n \to +\infty
\end{aligned}$$

where we used the non-expansiveness of $\mathcal{T}$. Hence, $\lim_n \mathcal{T}^{k'_n} v_\alpha = v_\alpha$ and $v_\alpha$ belongs to $\Omega(\mathcal{T})$. Consequently, we deduce that

$$\begin{aligned}
\|\mathcal{T}^{k'_n} v - v\|_\infty &= \|\mathcal{T}^{k'_n+k_n} v(-k_n,\cdot) - \mathcal{T}^{k_n} v(-k_n,\cdot)\|_\infty \\
&\le \|\mathcal{T}^{k'_n} v(-k_n,\cdot) - v(-k_n,\cdot)\|_\infty \\
&\le \|\mathcal{T}^{k'_n} v(-k_n,\cdot) - \mathcal{T}^{k'_n} v_\alpha\|_\infty + \|\mathcal{T}^{k'_n} v_\alpha - v_\alpha\|_\infty + \|v_\alpha - v(-k_n,\cdot)\|_\infty \\
&\le \|v(-k_n,\cdot) - v_\alpha\|_\infty + \|\mathcal{T}^{k'_n} v_\alpha - v_\alpha\|_\infty + \|v_\alpha - v(-k_n,\cdot)\|_\infty \quad \longrightarrow 0 \quad \text{as } n \to +\infty
\end{aligned}$$

Therefore, $\lim_n \mathcal{T}^{k'_n} v = v$ and $v$ belongs to $\Omega(\mathcal{T})$.

□

**Remark 6.4.3.** Note that this theorem implies that for a viscosity solution, being recurrent in positive times is equivalent to being recurrent in negative times.

# Chapitre 7

# Representation of the Non-Wandering set $\Omega(\mathcal{T})$ of the Lax-Oleinik Semigroup $\mathcal{T}$

The goal of this chapter is to establish that, in non-autonomous weak-KAM theory, the non-wandering set $\Omega(\mathcal{T})$ is more relevant to consider than the set of weak-KAM solutions $\mathrm{Fix}(\mathcal{T})$.

As an initial result supporting this viewpoint, we note that in the non-autonomous framework, P. Bernard and J.-M. Roquejoffre [BR04] demonstrated that calibrations on the Mather set apply to non-wandering viscosity solutions of the Hamilton-Jacobi equation 5.0.1. This key result implies that non-wandering viscosity solutions share significant properties with weak-KAM solutions, such as $C^{1,1}$-regularity when restricted to the Mather set and uniqueness on this set, in the sense that two non-wandering viscosity solutions that coincide at time 0 on the Mather set $\mathcal{M}_0$ coincide everywhere (see Theorem 7.2.1).

In his weak-KAM theory, A.Fathi explicited examples of weak-KAM solutions using the Peierls barrier $h^\infty(x,\cdot)$ and established various properties of weak-KAM solutions and G. Contreras [Con01] noticed that these are the main bric to express all weak-KAM solutions $u$ through a representation formula given as follows

$$u(x) = \inf_{y\in\mathbb{M}}\{\psi(y) + h^\infty(y,\cdot)\} \tag{7.0.1}$$

where $\mathbb{M}$ is a subset of the Mather set formed by representatives of static classes (See Definition 5.2.22) and the map $\psi : \mathcal{M} \to \mathbb{R}$ is dominated by the Peierls barrier $h^\infty$ (see Definition 7.1.1). This formula was later extended for non-autonomous Tonelli Hamiltonians by G. Contreras, R. Iturriaga, and H. Sánchez-Morgado in [CISM13]. Since $n$-periodic viscosity solutions can be regarded as weak-KAM solutions of the modified Hamiltonian $H_n = nH(nt, x, p)$, this representation formula extends naturally to the set $\mathrm{Per}_n(\mathcal{T})$ of $n$-periodic viscosity solutions.

In the autonomous framework, Fathi's convergence Theorem claims that all viscosity solutions converge to a weak-KAM solution, implying that $\Omega(\mathcal{T}) = \mathrm{Fix}(\mathcal{T})$, so that the non-wandering set is described by the representation formula. In the autonomous setting, Fathi's convergence theorem asserts that all viscosity solutions converge to a weak-KAM solution, implying that $\Omega(\mathcal{T}) = \mathrm{Fix}(\mathcal{T})$, and therefore, the non-wandering set can be described by the representation formula. However, this convergence theorem does not hold in the non-autonomous case [FM00]. Nevertheless, Bernard and Roquejoffre [BR04] proved that in dimension 1, there exists a positive integer $N > 0$ such that all viscosity solutions have $N$-periodic limit points, yielding $\Omega(\mathcal{T}) = \mathrm{Per}_N(\mathcal{T})$, and thereby obtaining a non-wandering set still characterized by a representation formula.

In Chapter 8, we will see that in higher dimensions, there exist Tonelli Hamiltonians for which $\Omega(\mathcal{T}) \neq \mathrm{Per}(\mathcal{T})$. This raises the question of whether these general non-wandering sets can still be represented by a modified representation formula. This chapter aims to address this question. We will define a generalized Peierls barrier that will play a role analogous to $h^\infty$ in a generalized representation formula for $\Omega(\mathcal{T})$.

Subsequently, We apply this representation formula to prove Fathi's convergence theorem for autonomous systems, confirming that $\Omega(\mathcal{T}) = \mathrm{Fix}(\mathcal{T})$. We also explicit the representation formula for $n$-periodic viscosity solutions in $\mathrm{Per}(\mathcal{T})$. Furthermore, we establish that the dynamics of non-wandering viscosity solutions are governed by the Lagrangian flow on the Mather set. Specifically, we show that if the Mather set consists solely of $N$-periodic orbits for some integer $N$, then all non-wandering viscosity solutions are $N$-periodic. Furthermore, we show that if the restriction of the Lagrangian flow to the Mather set is uniformly recurrent for a sequence $p_n$, then all non-wandering viscosity solutions are uniformly recurrent for the same sequence $p_n$.

## 7.1 Main Results of the Chapter

We focus on establishing a generalized representation formula on $\Omega(\mathcal{T})$. To achieve this, we first introduce the concept of domination, which is essential for developing representation formulas.

**Definition 7.1.1.** Let $X$ be a set and let $f : X \times X \to \mathbb{R}$ be a map. A map $\psi : X \to \mathbb{R}$ is said to be *$f$-dominated on $X$* if for all $x$ and $y$ in $X$, we have

$$\psi(y) - \psi(x) \leq f(x,y) \tag{7.1.1}$$

We denote by $\mathrm{Dom}(X, f)$ the set of $f$-dominated maps on $X$.

The goal is to identify suitable tame maps $f$ and sets $X$ where domination holds. Additionally, a uniqueness theorem must be valid on the set $X$, which means that if $u$ and $v$ are two elements of $\Omega(\mathcal{T})$ whose restrictions to $X$ are equal, then $u$ and $v$ themselves must be equal.

In the representation of weak-KAM solutions by Contreras, Iturriaga, and Sánchez-Morgado [CISM13], the set $X$ is chosen to be $\mathbb{M}$, which consists of the static classes mentioned in (7.0.1), and the map $f$ corresponds to the Peierls barrier $h^\infty$.

We will introduce a generalized Peierls barrier $\underline{k} : \mathcal{M}_0 \times M \to \mathbb{R}$, where $\mathcal{M}_0$ refers to the restriction of the Mather set at time $t = 0$ (see Definition 5.2.1 for $\mathcal{M}_0$ and Definition 7.3.8 for the barrier $\underline{k}$).

Besides, we define

- A pseudometric $\underline{d} : \mathcal{M}_0 \times \mathcal{M}_0 \to \mathbb{R}$ by

$$\underline{d}(x,y) = \underline{k}(x,y) + \underline{k}(y,x) \tag{7.1.2}$$

- An equivalence relation $\sim$ on $\mathcal{M}_0$ by

$$x \sim y \Longleftrightarrow \underline{d}(x,y) = 0 \tag{7.1.3}$$

- And the *generalized static classes* as the equivalence classes of the equivalence relation $\sim$.

We denote by $\underline{\mathbb{M}}$ the set of generalized static classes and assume that every element of $\underline{\mathbb{M}}$ is represented by an element of $\mathcal{M}_0$ so that we have the inclusion $\underline{\mathbb{M}} \subset \mathcal{M}_0$. Then, we obtain the following result.

**Theorem 7.1.2.** *We have the following bijection*

$$\begin{array}{rcl} \Psi_{\underline{k}} : \mathrm{Dom}(\underline{\mathbb{M}}, \underline{k}) & \longrightarrow & \Omega(\mathcal{T}) \\ \psi & \longmapsto & \inf\limits_{y \in \underline{\mathbb{M}}} \{\psi(y) + \underline{k}(y, \cdot)\} \end{array} \tag{7.1.4}$$

*with its inverse being the restriction map*

$$\begin{array}{rcl}\Phi_{\underline{k}} : \Omega(\mathcal{T}) & \longrightarrow & \mathrm{Dom}(\underline{\mathbb{M}}, \underline{k}) \\ v & \longmapsto & v_{|\underline{\mathbb{M}}}\end{array} \tag{7.1.5}$$

Finally, we will explore various applications of this representation formula. It is important to note that these applications may not be direct implications of the theorem but rather adaptations of its proofs and underlying ideas.

First, we will consider the autonomous framework and demonstrate Fathi's convergence theorem.

**Corollary 7.1.3.** *(Fathi's Convergence Theorem [Fat98]) Let $H : T^*M \to \mathbb{R}$ be an autonomous Tonelli Hamiltonian. For any initial data $u \in \mathcal{C}(M, \mathbb{R})$, the viscosity solution $u(t,x) = \mathcal{T}^t u(x)$ converges at $+\infty$ to a weak-KAM solution $v : M \to \mathbb{R}$ of*

$$H(x, d_x u) = \alpha_0 \tag{7.1.6}$$

Another application involves representing the set $\mathrm{Per}_n(\mathcal{T})$ of $n$-periodic viscosity solutions for a fixed integer period $n$, where $n$ does not necessarily have to be the minimal period. In this context, we will define the $n$-Peierls barrier $h^{n\infty}$, as first introduced by A. Fathi and J. N. Mather in [FM00] (see Definition 7.4.3).

Similarly to the general case, we define

- A pseudometric $d_n : \mathcal{M}_0 \times \mathcal{M}_0 \to \mathbb{R}$ by

$$d_n(x,y) = h^{n\infty}(x,y) + h^{n\infty}(y,x) \tag{7.1.7}$$

- An equivalence relation $\sim_n$ on $\mathcal{M}_0$ by

$$x \sim y \Longleftrightarrow d_n(x,y) = 0 \tag{7.1.8}$$

  which equivalence classes are called $n$-static classes and are represented by a subset $\mathbb{M}_n$ of the Mather set $\mathcal{M}_0$.

In this context, the set $\mathbb{M}_n$ serves as a uniqueness set for $n$-periodic viscosity solutions. Specifically, if two $n$-periodic viscosity solutions $u$ and $v$ in $\mathrm{Per}_n(\mathcal{T})$ coincide on $\mathbb{M}_n$, then they coincide everywhere. Furthermore, we establish that every $n$-periodic element of $\mathrm{Per}_n(\mathcal{T})$ is $h^{n\infty}$-dominated on $\mathbb{M}_n$. Consequently, we obtain the following result :

**Theorem 7.1.4.** *We have the following bijection*

$$\begin{array}{rcl}\Psi_n : \mathrm{Dom}(\mathbb{M}_n, h^{n\infty}) & \longrightarrow & \mathrm{Per}_n(\mathcal{T}) \\ \psi & \longmapsto & \inf\limits_{y \in \mathbb{M}_n} \{\psi(y) + h^{n\infty}(y, \cdot)\}\end{array} \tag{7.1.9}$$

*with its inverse being the restriction map*

$$\begin{array}{rcl} \Phi_n : \mathrm{Per}_n(\mathcal{T}) & \longrightarrow & \mathrm{Dom}(\mathbb{M}_n, h^{n\infty}) \\ v & \longmapsto & v_{|\mathbb{M}_n} \end{array} \tag{7.1.10}$$

Another implication of the representation Theorem 7.1.2 is that the dynamics on the non-wandering set $\Omega(\mathcal{T})$ can be controlled by the dynamics of the Lagrangian flow $\phi_L$ (see Section 5.1.1) on the Mather set. Based on this, we can establish the following two results :

**Corollary 7.1.5.** *If there exists a positive integer $N \geq 1$ such that $\phi^N_{L|\mathcal{M}_0} = Id_{\mathcal{M}_0}$, then $\Omega(\mathcal{T}) = \mathrm{Per}_N(\mathcal{T})$.*

**Corollary 7.1.6.** *If there exists an increasing sequence of positive integers $\underline{p} = (p_n)_{n\geq 0}$ such that $\phi^{-p_n}_{L|\tilde{\mathcal{M}}_0}$ uniformly converges to the identity, then the elements $v$ of $\Omega(\mathcal{T})$ are $\underline{p}$-recurrent i.e $\lim_n \mathcal{T}^{p_n} v = v$ with a uniform convergence on $v$.*

Section 7.2 provides a proof of the uniqueness theorem for non-wandering viscosity solutions on the Mather set $\mathcal{M}_0$, following [BR04]. Here, the calibration on the Mather set is demonstrated using a classical approach, inspired by A. Fathi's method in the autonomous case. In Section 7.3, we define generalized Peierls barriers, discuss their properties, and then present the proof of the main result, Theorem 7.1.2. Finally, Section 7.4 is dedicated to exploring various examples and proving the corollaries stated in the introduction.

## 7.2 The Uniqueness Theorem in $\Omega(\mathcal{T})$

This section is dedicated to proving a Uniqueness Theorem 7.2.4 for non-wandering viscosity solutions on the Mather set. This theorem will play a central role in the various representation formulas presented in the next sections, allowing to determine non-wandering viscosity solutions based on their restrictions to the Mather set.

This theorem was initially established by A. Fathi for weak-KAM solutions $\mathrm{Fix}(\mathcal{T})$ in the autonomous framework (see [Fat08]) and by M. Zavidovique for the discrete case (see [Zav23]). Subsequently, P. Bernard and J.-M. Roquejoffre extended it to non-wandering viscosity solutions in [BR04]. We offer an alternative proof of their version, which is a more classical approach adapted from A. Fathi's original proof.

### 7.2.1 Calibration on the Mather Set $\mathcal{M}$

**Proposition 7.2.1.** *Let $x$ be an element of the Mather set $\mathcal{M}_0$ with lift $\tilde{x}$ in $\tilde{\mathcal{M}}_0$ and let $\gamma : \mathbb{R} \to M$ be the projection on $M$ of the Lagrangian flow at $\tilde{x}$ i.e. $\gamma(t) = \pi \circ \phi^t_L(\tilde{x})$. Then every recurrent viscosity solution $v$ in $\Omega(\mathcal{T})$ is calibrated by the curve $\gamma$.*

*Proof.* Let $v$ be an element $\Omega(\mathcal{T})$. Let $k_n$ be a sequence of integers such that $v(k_n,\cdot)$ converges to $v$ in $\mathcal{C}(M,\mathbb{R})$. We know from the definition of viscosity solutions that $\mathcal{T}^t v = v(t,\cdot)$. Thus, for all real time $t$ and all point $\tilde{y}$ of $TM$, the definition of the operator $\mathcal{T}$ gives

$$v\big(t+k_n,\pi\circ\phi_L^{t,t+k_n}(\tilde{y})\big)-v\big(t,\pi\circ\tilde{y}\big) \leq A_L(\pi\circ\phi_L^{t,\tau}(\tilde{y})) = \int_t^{t+k_n}\Big(L(\tau,\phi_L^{t,\tau}(\tilde{y}))+\alpha_0\Big)\,d\tau \tag{7.2.1}$$

Let $\mu$ be a minimizing measure on $\mathbb{T}^1\times TM$ that has $\tilde{x}$ in its support $\mathrm{Supp}(\mu)$. We keep the same notation $\mu$ for its time-one periodic lift $\mu$ to $\mathbb{R}\times TM$. We integrate (7.2.1) in $(t,\tilde{y})\in[0,1]\times TM$ with respect to the lift $\mu$.

$$\begin{aligned}\int_0^1\int_{TM} v\big(t+k_n,\pi\circ\phi_L^{t,t+k_n}(\tilde{y})\big)\,d\mu - \int_0^1\int_{TM} v(t,\pi(\tilde{y}))\,d\mu&\\ \leq \int_0^1\int_{TM}\int_t^{t+k_n}\Big(L(\tau,\phi_L^{t,\tau}(\tilde{y}))+\alpha_0\Big)\,d\tau\,d\mu&\end{aligned} \tag{7.2.2}$$

We compute of the right-hand side while taking in consideration the time periodicity of the Lagrangian $L$ and the $\Phi_L^\tau$-invariance of $\mu$ where $\Phi_L^{k_n}$ has been defined in (5.2.5)

$$\begin{aligned}\int_0^1\int_{TM}\int_t^{t+k_n}\Big(L(\tau,\phi_L^{t,\tau}(\tilde{y}))+\alpha_0\Big)\,d\tau\,d\mu &= \int_0^1\int_{TM}\int_0^{k_n}\Big(L(t+\tau,\phi_L^{t,t+\tau}(\tilde{y}))+\alpha_0\Big)\,d\tau\,d\mu\\ &= \int_0^{k_n}\int_0^1\int_{TM}\Big(L(t+\tau,\phi_L^{t,t+\tau}(\tilde{y}))+\alpha_0\Big)\,d\mu\,d\tau\\ &= \int_0^{k_n}\Big(\int_0^1\int_{TM}L(t,\tilde{y})\,d\mu\;+\alpha_0\Big)\,d\tau = 0\end{aligned} \tag{7.2.3}$$

We shift our focus to the left hand side of (7.2.2). The $\Phi_L^{k_n}$-invariance of $\mu$ results in

$$\begin{aligned}\int_0^1\int_{TM} v\big(t+k_n,\pi\circ\phi_L^{t,t+k_n}(\tilde{y})\big)\,d\mu &= \int_{k_n}^{k_n+1}\int_{TM} v(t,\pi(\tilde{y}))\,d\mu\\ &= \int_0^1\int_{TM} v(t+k_n,\pi(\tilde{y}))\,d\mu\end{aligned}$$

Thus

$$\begin{aligned}\int_0^1\int_{TM} v\big(t+k_n,\pi\circ\phi_L^{t,t+k_n}(\tilde{y})\big)\,d\mu - \int_0^1\int_{TM} v(t,\pi(\tilde{y}))\,d\mu&\\ = \int_0^1\int_{TM} v(t+k_n,\pi(\tilde{y}))\,d\mu - \int_0^1\int_{TM} v(t,\pi(\tilde{y}))\,d\mu&\\ = \int_0^1\int_{TM} v(k_n+t,\pi(\tilde{y}))-v(t,\pi(\tilde{y}))\,d\mu&\end{aligned} \tag{7.2.4}$$

However, we know from Proposition 6.4.2 that the restrictions $v(t+k_n,x)$ uniformly converge to $v$ on $\mathbb{R}\times M$. Hence, we deduce that

$$\lim_n \int_0^1 \int_{TM} v(k_n + t, \pi(\tilde{y})) - v(t, \pi(\tilde{y}))\, d\mu = 0 \tag{7.2.5}$$

Now, Gathering (7.2.2), (7.2.3), (7.2.4) and (7.2.5), and if we define the defect of calibration $\delta_{k_n}(t,\tilde{y})$ by

$$\delta_{k_n}(t,\tilde{y}) = \alpha_0.k_n + \int_t^{t+k_n} L(\tau, \phi_L^{t,\tau}(\tilde{y}))\, d\tau - \left[v\big(t,\pi\circ\tilde{y}\big) - v\big(t,\pi\circ\tilde{y}\big)\right] \geq 0 \tag{7.2.6}$$

we get

$$\lim_n \int_0^1 \int_{TM} \delta_{k_n}(t,\tilde{y})\, d\mu = 0$$

where the integrand is non-negative and increasing in $n$. This implies that for $\mu$-almost all $(t,\tilde{y})$ in $\mathrm{Supp}(\mu)$, we have for all $s > 0$,

$$\delta_s(t,\tilde{y}) = \alpha_0.s + \int_t^{t+s} L(\tau, \phi_L^{t,\tau}(\tilde{y}))\, d\tau - \left[v\big(t,\pi\circ\tilde{y}\big) - v\big(t,\pi\circ\tilde{y}\big)\right] = 0$$

And by continuity of $L$, $v$ and hence $\delta_s$, the equality extends to $\mathrm{Supp}(\mu)$. Since $\mu$ is invariant by the Lagrangian flow $\phi_L$ and $\tilde{x}$ belongs to its support, we infer that the graph of the curve $(t,\gamma(t))$ also belongs to $\mathrm{Supp}(\mu)$ so that for all negative integer $m \leq 0$, $(m,\gamma(m)$ also belongs to $\mathrm{Supp}(\mu)$ and the associated curve are calibrated by all non-wandering viscosity solutions in $\Omega(\mathcal{T})$. Moreover, by Proposition 6.4.2, we have $v(m+k_n,\cdot)$ converges to $v(m,\cdot)$ and in particular, $v(m,\cdot)$ belongs to $\Omega(\mathcal{T})$ and it calibrates by $\gamma(t+m)$. Therefore, $\gamma$ is calibrated by $v$ on $\mathbb{R}$. □

**Remark 7.2.2.** The application of Theorem 5.1.20 to the curves $\gamma$ of $\mathcal{M}$ reveals that the Mather set (and more generally, the Peierls set $\mathcal{A}_0$ defined below) is a set of differentiability for all non-wandering viscosity solutions.

### 7.2.2 The Uniqueness Theorem

**Lemma 7.2.3.** *Let $u$ be in $\Omega(\mathcal{T})$ with corresponding global viscosity solution $u(t,x)$ : $\mathbb{R}\times M \to \mathbb{R}$. Let $x$ be an element of $M$ and $\gamma : (-\infty, 0] \to M$ with $\gamma(0) = x$ be a $u$-calibrated curve. Then there exists an increasing sequence of integers $k_n$ such that $\gamma(-k_n)$ converges to a point $x_\alpha$ of $\mathcal{M}_0$.*

*Proof.* Using Riesz representation theorem, we define for all positive time $t > 0$ a Borel probability measure $\mu_t : \mathcal{C}(\mathbb{T}^1 \times TM, \mathbb{R}) \to \mathbb{R}$ by

$$\mu_t(\theta) = \frac{1}{t}\int_{-t}^0 \theta(s,\gamma(s),\dot{\gamma}(s))\, ds$$

We know from remark 5.1.16 that $\gamma$ is minimizing. Hence, the A priori compactness of minimizing curves implies that $(\gamma, \dot\gamma)$ belongs to a compact subset $K$ of $TM$, so that $\text{supp}(\mu_t) \subset K$ is compact. Consequently, we can extract an increasing sequence of integers $k_n$ such that $\mu_{k_n}$ weak-$*$ converges to a compactly supported probability measure $\mu$. Additionally, since the curve $\gamma$ is minimizing, it follows the Lagrangian flow $\phi_L$ and we deduce that the measure $\mu$ is $\Phi_L$-invariant.

Weak-$*$ convergence applied to $\theta = L$ gives

$$\int_{\mathbb{T}^1\times TM} L\, d\mu = \lim_n \int_{\mathbb{T}^1\times TM} L\, d\mu_{k_n} = \lim_n \frac{1}{k_n}\int_{-k_n}^0 L(s,\gamma(s),\dot\gamma(s))\, ds$$

Addtionally, the calibration of $\gamma$ yields

$$\frac{1}{k_n}\int_{-k_n}^0 L(s,\gamma(s),\dot\gamma(s))\, ds + \alpha_0 = \frac{1}{k_n}\big[u(0,x) - u(-k_n,\gamma(-k_n))\big]$$

However, Theorem 6.0.2 claims that $u$ is bounded. Thus, we deduce that

$$\int_{\mathbb{T}\times TM} L\, d\mu + \alpha_0 = \lim_n \frac{1}{k_n}\big[u(0,x) - u(-k_n,\gamma(-k_n))\big] = 0$$

and the measure $\mu$ is minimizing. Therefore, by definition of the Mather set $\tilde{\mathcal{M}}$, we get the inclusion

$$\text{supp}(\mu) \subset \tilde{\mathcal{M}}$$

We now show that $\text{Supp}(\mu)$ belongs to the $\alpha$-limit of the curve $\gamma$. Let $(0,\tilde x) = (0,x,v)$ be an element of $\text{supp}(\mu)$ and for all $m \geq 1$, let $A_m$ and $B_m$ be the balls of center $(0,\tilde x)$ and respective radii $1/m$ and $1/(2m)$ in some chart of $\mathbb{T}^1\times TM$. We consider a continuous bump map $\chi_m : \mathbb{T}^1\times TM \to [0,1]$ supported on $A_m$ and equal to 1 on $B_m$. Then, if we denote by $\chi_{A_m}$ the indicator function of the set $A_m$, we have $\chi_{B_m} \leq \chi_m \leq \chi_{A_m}$ and

$$\mu_{k_n}(B_m) \leq \mu_{k_n}(\chi_m) \leq \mu_{k_n}(A_m) \quad \text{and} \quad 0 < \mu(B_m) \leq \mu(\chi_m) \leq \mu(A_m)$$

where the positivity of $\mu(B_m)$ is due to the fact that $B_m$ is a neighbourhood of $(0,\tilde x) \in \text{Supp}(\mu)$. And by definition of the weak-$*$ convergence, we obtain

$$0 < \mu(\chi_m) = \lim_n \mu_{k_n}(\chi_m) \leq \liminf_n \mu_{k_n}(A_m) \tag{7.2.7}$$

Moreover, we have

$$\begin{aligned}\mu_{k_n}(A_m) &= \frac{1}{k_n}\int_{-k_n}^0 \chi_{A_m}(s,\gamma(s),\dot\gamma(s))\, ds \\ &= \frac{1}{k_n}\text{Leb}\big(\{s\in[-k_n,0] \mid (s,\gamma(s),\dot\gamma(s)) \in A_m\}\big)\end{aligned}$$

where Leb stands for the Lebesgue measure. We consider the real number $t_n \in \mathbb{R}$ defined by

$$t_n := \inf\left\{\tau \in [0, k_n] \,\middle|\, \{s \in [-k_n, 0] \mid (s, \gamma(s), \dot{\gamma}(s)) \in A_m\} \subset [-\tau, 0]\right\}$$

We have $(-t_n, \gamma(-t_n), \dot{\gamma}(-t_n))$ belongs to the closure $\overline{A}_m$ of $A_m$, and

$$\mu_{k_n}(A_m) \leq \frac{1}{k_n} \mathrm{Leb}([-t_n, 0]) = \frac{t_n}{k_n}$$

Thus, we infer from the inequalities (7.2.7) that

$$0 < \mu(\chi_m) \leq \liminf_n \frac{t_n}{k_n}$$

and since $k_n$ diverges to $+\infty$, we deduce that $\lim_n t_n = +\infty$.

For all $m \geq 1$, we constructed a sequence $t_n$ such that

$$\lim_n t_n = +\infty \quad \text{and} \quad (-t_n - \lfloor -t_n \rfloor, \gamma(-t_n), \dot{\gamma}(-t_n)) \in \overline{A}_m \subset \mathbb{T}^1 \times TM \tag{7.2.8}$$

By extracting, we get two increasing sequences $m_i$ and $t_{n_i}$ verifying (7.2.8). And since $m_i$ is increasing and $A_m$ is of radius $1/m_i$, we obtain

$$\lim_i \,(-t_{n_i} - \lfloor -t_{n_i} \rfloor, \gamma(-t_{n_i}), \dot{\gamma}(-t_{n_i})) = (0, \tilde{x})$$

and in particular

$$\lim_i \,(-t_{n_i} - \lfloor -t_{n_i} \rfloor, \gamma(-t_{n_i})) = (0, x) \in \pi(\mathrm{supp}(\mu)) \cap \big(\{0\} \times M\big) \subset \mathcal{M}_0$$

In order to obtain integer times instead of $t_{n_i}$, we recall that $(\gamma, \dot{\gamma})$ belongs to a compact set $K$ of $TM$ and it follows the Lagrangian flow $\phi_L$ so that $\phi_L^{-\lfloor -t_{n_i} \rfloor, -t_{n_i}} = \phi_L^{0, -t_{n_i} - \lfloor -t_{n_i} \rfloor}$ restricted to $K$ converges uniformly to the identity and

$$\lim_i \,(\gamma(\lfloor -t_{n_i} \rfloor), \dot{\gamma}(\lfloor -t_{n_i} \rfloor)) = \lim_i \left(\phi_L^{0, -t_{n_i} - \lfloor -t_{n_i} \rfloor}\right)^{-1} (\gamma(-t_{n_i}), \dot{\gamma}(-t_{n_i})) = id(\tilde{x}) = \tilde{x}$$

$\square$

**Theorem 7.2.4.** *Let $u$ and $v$ be two non-wandering viscosity solutions in $\Omega(\mathcal{T})$ such that $u_{|\mathcal{M}_0} = v_{|\mathcal{M}_0}$. Then $u = v$ everywhere.*

*Proof.* Let $(t, x)$ be a fixed element of $\mathbb{T}^1 \times M$ and $\gamma : (-\infty, t] \to M$ with $\gamma(t) = x$ be a curve calibrated by $v$ given by proposition 5.1.17. For all times $s_1 < s_2$, we have

$$v(s_2, \gamma(s_2)) - v(s_1, \gamma(s_1)) = \int_{s_1}^{s_2} L(\tau, \gamma(\tau), \dot{\gamma}(\tau))\, d\tau + \alpha_0.(s_2 - s_1) \tag{7.2.9}$$

and from the definition of viscosity solutions

$$u(s_2,\gamma(s_2)) - u(s_1,\gamma(s_1)) \le \int_{s_1}^{s_2} L(\tau,\gamma(\tau),\dot\gamma(\tau))\,d\tau + \alpha_0.(s_2 - s_1)$$

Replacing the right hand side by (7.2.9), we get

$$u(s_2,\gamma(s_2)) - u(s_1,\gamma(s_1)) \le v(s_2,\gamma(s_2)) - v(s_1,\gamma(s_1))$$

and more precisely

$$\big(u - v\big)(s_2,\gamma(s_2)) \le \big(u - v\big)(s_1,\gamma(s_1))$$

Hence $(u-v)(s,\gamma(s))$ is non-increasing in time $s$. Since $u$ and $v$ are bounded by Theorem 6.0.2, it follows that $\big(u-v\big)(s,\gamma(s))$ has a finite limit $l$ at $-\infty$ that we will determine.

Now apply Lemma 7.2.3 to the curve $\gamma : \big(-\infty, \lfloor t\rfloor\big] \to M$ to get a an increasing sequence of integers $k_n$ and an element $x_\alpha$ of $\mathcal{M}_0$ such that $\gamma(-k_n)$ converges to $x_\alpha$. We are interested in computing

$$l = \lim_{s\to-\infty} u(s,\gamma(s)) - v(s,\gamma(s)) = \lim_n u(-k_n,\gamma(-k_n)) - v(-k_n,\gamma(-k_n))$$

The equicontinuity of the families $\big(u(t,\cdot)\big)_t$ and $\big(v(t,\cdot)\big)_t$ given by Corollary 5.1.14 allow to replace $\gamma(-k_n)$ by its limit $x_\alpha$ to get

$$l = \lim_n u(-k_n,x_\alpha) - v(-k_n,x_\alpha)$$

Set $x_\alpha(t) = \pi\circ\phi_L^{-k_n,t}(\tilde x_\alpha)$ where $\tilde x_\alpha \in \tilde{\mathcal{M}}_0$ is the lift of $x_\alpha$. Proposition 7.2.1 implies that $x_\alpha(t)$ is calibrated by $u$ and $v$, and

$$u(0,x_\alpha(0)) - u(-k_n,x_\alpha) = \int_{-k_n}^{0} L(s,x_\alpha(s)\dot{x_\alpha}(s))\,ds + \alpha_0.k_n = v(0,x_\alpha(0)) - v(-k_n,x_\alpha)$$

and since $x_\alpha(0)$ belongs to $\mathcal{M}_0$ and $u_{|\mathcal{M}_0} = v_{|\mathcal{M}_0}$, we obtain

$$u(-k_n,x_\alpha) - v(-k_n,x_\alpha) = u(0,x_\alpha) - v(0,x_\alpha) = 0$$

We proved that $(u-v)(s,\gamma(s))$ is non-increasing in time $s$ and has a null limit at $-\infty$. Therefore, $(u-v)(s,\gamma(s))$ is non-negative and

$$u(t,x) = u(t,\gamma(t)) \le v(t,\gamma(t)) = v(t,x)$$

The symmetry between $u$ and $v$ gives the inverse inequality and we conclude that $u(t,x) = v(t,x)$ for all $(t,x)$ in $\mathbb{T}^1\times M$. $\square$

## 7.3 Representation Formulas for the Non-Wandering Set $\Omega(\mathcal{T})$

This is the central section of the chapter, divided into two subsections. The first subsection introduces various Peierls barriers, which will be used to construct an initial, non-canonical representation formula that can be easily adapted to explicit examples. The second subsection is dedicated to the introduction of the generalized Peierls barrier $\underline{k}$ mentioned in the introduction, and to the proof of the main representation Theorem 7.1.2 of this chapter.

### 7.3.1 Representation Formula on $\mathcal{M}_0^R$

In this subsection, we establish a non-canonical representation formula. To achieve this, we first introduce the $\underline{p}$-Peierls barriers associated with increasing positive integer sequences $\underline{p} = (p_n)_n$, and use them to construct a non-canonical barrier $\underline{h}$, defined on the recurrent Mather set $\mathcal{M}_0^R$.

#### The $\underline{p}$-Peierls Barrier

We define the $\underline{p}$-Peierls barriers, which yield a wide range of non-wandering viscosity solutions.

**Definition 7.3.1.** 1. For any increasing sequence $\underline{p} = (p_n)_{n\geq 0}$ in $\mathbb{N}$, we define the *$\underline{p}$-Peierls Barrier* $h^{\underline{p}} : M \times M \to \mathbb{R}$ by

$$h^{\underline{p}}(x,y) = \liminf_{n} h^{p_n}(x,y) \tag{7.3.1}$$

with the corresponding time-dependant $\underline{p}$-barrier

$$h^{\underline{p}+t}(x,y) = h^{\underline{p}}(t,x,y) = \liminf_{n} h^{t+p_n}(x,y) \tag{7.3.2}$$

where $h^t$ is the potential explicited in (5.2.9).

More generally, for all two times $s$ and $t$, we define

$$h^{s,\underline{p}+t}(x,y) = \liminf_{n\to\infty} h^{s,p_n+t}(x,y) \tag{7.3.3}$$

2. The *Peierls Barrier* $h^{\infty} : M \times M \to \mathbb{R}$ is the $\underline{p}$-barrier for $p_n = n$ defined by

$$h^{\infty}(x,y) = \liminf_{n\to\infty} h^{n}(x,y) \tag{7.3.4}$$

**Proposition 7.3.2.** *1. (Finiteness) For all $(t, x, y) \in \mathbb{R} \times M \times M$, the Peierls barrier $h^{\underline{p}}(t, x, y)$ is finite.*

*2. (Regularity) The Peierls barrier $h^{\underline{p}}$ is $\kappa_\varepsilon$-Lipschitz for all $\varepsilon > 0$ with $\kappa_\varepsilon$ being the Lipschitz constant introduced in Proposition 5.1.13.*

*3. (Viscosity Solution) For all $x \in M$, $h^{\underline{p}}(,\cdot, x, \cdot)$ is a viscosity solution of the Hamilton-Jacobi equation (5.0.1).*

*4. (Liminf Property) For all points $x$ and $y$ in $M$ and all sequences of points $(x_n)_n$ and $(y_n)_n$ in $M$ respectively converging to $x$ and $y$, we have*

$$h^{\underline{p}}(x, y) = \liminf_n h^{p_n}(x_n, y_n) \tag{7.3.5}$$

*Proof.* 1. Direct consequence of Proposition 5.2.10.

2. Direct consequence of Proposition 5.1.13.

3. For all $x \in M$, we know from Proposition 5.2.7 that $h^{p_n}(\cdot, x, \cdot)$ is a viscosity solution. Hence, we deduce from Proposition 6.2.3 that $h^{\underline{p}}(\cdot, x, \cdot) = \liminf_n h^{p_n}(\cdot, x, \cdot)$ is also a viscosity solution.

4. We have from the regularity of $h$ that for all integers $n$ and $k \geq 1$

$$|h^k(x, y) - h^k(x_n, y_n)| \leq \kappa_1.\big(d(x, x_n) + d(y, y_n)\big)$$

Hence, setting $k = p_n$ and taking the liminf on $n$, we get

$$h^{\underline{p}}(x, y) = \liminf_n h^{p_n}(x, y) = \liminf_n h^{p_n}(x_n, y_n)$$

□

### The Barrier $\underline{h}$

We start by introducing a new barrier $\underline{h}$ based on the $\underline{p}$-Peierls barriers. For that purpose, we consider for every point $x$ of the recurrent Mather set $\mathcal{M}_0^R$ with lift $\tilde{x}$ in $\tilde{\mathcal{M}}_0^R$ an increasing sequence $\underline{p}^x = (p_n^x)_{n \geq 0}$ of $\mathbb{N}$ such that $\tilde{x}$ is $-\underline{p}^x$-recurrent by the Lagrangian flow $\phi_L$.

**Definition 7.3.3.** 1. We define the barrier $\underline{h} : \mathcal{M}_0^R \times M \to \mathbb{R}$ by

$$\underline{h}(x, y) = h^{\underline{p}^x}(x, y) = \liminf_n h^{p_n^x}(x, y) \tag{7.3.6}$$

with the corresponding time-depending barrier

$$\underline{h}^t(x, y) = \underline{h}(t, x, y) = h^{\underline{p}^x + t}(x, y) \tag{7.3.7}$$

**Remark 7.3.4.** 1. Note that, in general, this barrier $\underline{h}(x, y)$ is not continuous on $x$.

2. The definition depends heavily on the choice of the sequences $\underline{p}^x$, which is why $\underline{h}$ is not canonical. This issue will be addressed in the next subsection about representation formulas on $\mathcal{M}_0$.

**Proposition 7.3.5.** *1. For all $x \in \mathcal{M}_0^R$, the map $\underline{h}(\cdot, x, \cdot)$ is a viscosity solution of the Hamilton-Jacobi equation (5.0.1).*

*2. For all $x \in \mathcal{M}_0^R$, the map $\underline{h}(\cdot, x, \cdot) : \mathbb{R} \times M \to \mathbb{R}$ is $\kappa_\varepsilon$-Lipschitz for all $\varepsilon > 0$ with $\kappa_\varepsilon$ being the Lipschitz constant introduced in Proposition 5.1.13.*

*3. For all $x$ in $\mathcal{M}_0^R$, we have $\underline{h}(x, x) = h^{\underline{p}^x}(x, x) = 0$.*

*4. For all $x \in \mathcal{M}_0^R$, the viscosity solution $\underline{h}(\cdot, x, \cdot)$ is $\underline{p}^x$-recurrent and*

$$\|\underline{h}^{p_n^x}(x, \cdot) - \underline{h}(x, \cdot)\|_\infty \leq 2\kappa_1.d(x(-p_n^x), x) \tag{7.3.8}$$

*Proof.* 1 and 2. Since $\underline{h}(x, \cdot) = h^{\underline{p}^x}(x, \cdot) = \liminf_n h^{p_n^x}(x, \cdot)$, these properties follow immediately from Proposition 7.3.2.

3. Consider the point $\tilde{x} \in \tilde{\mathcal{M}}_0$ that projects to $x$ on $M$ and set $x(t)$ to be the curve $x(t) = \pi \circ \phi_L^t(\tilde{x})$. We have by definition of the sequence $p_n^x$ that $x(-p_n^x)$ converges to $x$. Then, we get

$$h^{\underline{p}^x}(x, x) = \liminf_n h^{p_n^x}(x, x) = \liminf_n h^{p_n^x}(x(-p_n^x), x)$$

where we used the liminf property (7.3.5) of the Peierls barrier and the fact that $x(t)$ is minimizing seen in Remark 5.2.5. Let $u$ be any weak-KAM solution. We know from proposition 7.2.1 that $x(t)$ is calibrated by $u$, then by continuity and 1-time periodicity of $u$, we get

$$h^{p_n^x}(x(-p_n^x), x) = u(x) - u(-p_n^x, x(-p_n^x)) = u(x) - u(x(-p_n^x)) \longrightarrow 0 \quad \text{as } n \to \infty$$

Therefore, we deduce that

$$h^{\underline{p}^x}(x, x) = \lim_n h^{p_n^x}(x(-p_n^x), x) = 0$$

4. The fact that $h^{\underline{p}}(y, \cdot)$ is recurrent follows from the fact that it is a bounded global viscosity solution. This is due to Proposition 5.2.10 and the recurrence follows from Theorem 6.0.2. To demonstrate $\underline{p}$-recurrence, we will need to prove the identity (7.3.8).

Let $y$ be a fixed point of $\mathcal{M}_0^R$. Fix an integers $n$ and let $(n_i = n_i(n))_{i \geq 0}$ be an increasing sequence of integers depending on $n$ such that $h^{\underline{p}^y + p_n^y}(y, x) = \lim_i h^{p_{n_i}^y + p_n^y}(y, x)$. Now fix an integer $i \geq 0$ and let $k_0 = k_0(n, i)$ be an integer such that for all $k \geq k_0$, we get $p_k^y > p_{n_i}^y + p_n^y$.

For such $k \geq k_0$, we have

$$h^{p_k^y}(y,x) \leq h^{p_k^y - p_{n_i}^y - p_n^y}(y,y) + h^{p_{n_i}^y + p_n^y}(y,x) \tag{7.3.9}$$

Let $\tilde{y}$ be the lift of point of $y$ in the Mather set $\tilde{\mathcal{M}}_0$ and consider the curve $y(t) = \pi \circ \phi_L^t(\tilde{y})$. We know from the regularity Proposition 5.1.13 on the potential $h$ that for all integer $q \geq 1$

$$|h^q(y(-p_k^y), y(-p_{n_i}^y - p_n^y)) - h^q(y,y)| \leq \kappa_1.[d(y(-p_k^y), y) + d(y(-p_{n_i}^y - p_n^y), y)]$$

So that for $q = p_k^y - p_{n_i}^y - p_n^y$, we get

$$|h^{p_k^y - p_{n_i}^y - p_n^y}((y(-p_k^y), y(-p_{n_i}^y - p_n^y)) - h^{p_k^y - p_{n_i}^y - p_n^y}(y,y)| \leq \kappa_1.[d(y(-p_k^y), y) + d(y(-p_{n_i}^y - p_n^y), y)] \tag{7.3.10}$$

Let $v$ be a weak-KAM solution. We know from Proposition 7.2.1 that $y(t)$ is calibrated by $v$ and from Proposition 5.1.13 that $v = \mathcal{T}^1 v$ is $\kappa_1$-Lipschitz, hence

$$|h^{p_k^y - p_{n_i}^y - p_n^y}(y(-p_k^y), y(-p_{n_i}^y - p_n^y))| = |v(y(-p_{n_i}^y - p_n^y)) - v(y(-p_k^y))| \leq \kappa_1.d(y(-p_{n_i}^y - p_n^y), y(-p_k^y)) \tag{7.3.11}$$

Hence, we deduce from (7.3.10) that

$$\begin{aligned} h^{p_k^y - p_{n_i}^y - p_n^y}(y,y) &\leq h^{p_k^y - p_{n_i}^y - p_n^y}((y(-p_k^y), y(-p_{n_i}^y - p_n^y)) + \kappa_1.[d(y(-p_k^y), y) + d(y(-p_{n_i}^y - p_n^y), y)] \\ &\leq \kappa_1.[d(y(-p_{n_i}^y - p_n^y), y(-p_k^y)) + d(y(-p_k^y), y) + d(y(-p_{n_i}^y - p_n^y), y)] \end{aligned}$$

and from (7.3.9) that

$$h^{p_k^y}(y,x) \leq \kappa_1.[d(y(-p_{n_i}^y - p_n^y), y(-p_k^y)) + d(y(-p_k^y), y) + d(y(-p_{n_i}^y - p_n^y), y)] + h^{p_{n_i}^y + p_n^y}(y,x)$$

Let $k_i$ be an increasing sequence such that $k_i \geq k_0(n,i)$, i.e $p_{k_i}^y > p_{n_i}^y + p_n^y$, we obtain

$$\lim_i d(y(-p_{n_i}^y - p_n^y), y(-p_{k_i}^y)) = d(y(-p_n^y), y)$$

Then, taking the liminf on $i$ yields

$$\begin{aligned} h^{\underline{p}^y}(y,x) &\leq \kappa_1. \varliminf_i [d(y(-p_{n_i}^y - p_n^y), y(-p_{k_i}^y)) + d(y(-p_{k_i}^y), y) + d(y(-p_{n_i}^y - p_n^y), y)] + \varliminf_i h^{p_{n_i}^y + p_n^y}(y,x) \\ &= 2\kappa_1.d(y(-p_n^y), y) + h^{\underline{p}^y + p_n^y}(y,x) \end{aligned} \tag{7.3.12}$$

For the inverse inequality. Fix two integers $n$ and $k \geq 0$. The triangular inequality

(5.2.10) gives

$$h^{p_n^y+p_k^y}(y,x) \leq h^{p_n^y}(y,y) + h^{p_k^y}(y,x)$$

Moreover, we know from the Lipschitz regularity of the potential $h$ that

$$|h^{p_n^y}(y,y) - h^{p_n^y}(y(-p_n^y),y)| \leq \kappa_1.d(y(-p_n^y),y)$$

Bounding $h^{p_n^y}(y(-p_n^y),y)$ as in (7.3.11), we get

$$\begin{aligned} h^{p_n^y+p_k^y}(y,x) &\leq h^{p_n^y}(y(-p_n^y),y) + \kappa_1.d(y(-p_n^y),y) + h^{p_k^y}(y,x) \\ &\leq 2\kappa_1.d(y(-p_n^y),y) + h^{p_k^y}(y,x) \end{aligned}$$

Then, we Take the liminf on $k$ to obtain the desired inequality

$$h^{\underline{p}^y+p_n^y}(y,x) \leq 2\kappa_1.d(y(-p_n^y),y) + h^{\underline{p}^y}(y,x) \tag{7.3.13}$$

Gathering the inequalities (7.3.13) and (7.3.12) leads to

$$\|\underline{h}^{p_n^y}(y,\cdot) - \underline{h}(y,\cdot)\|_\infty \leq 2\kappa_1.d(y(-p_n^y),y) \longrightarrow 0 \quad \text{as } n \to +\infty \tag{7.3.14}$$

Therefore, $\underline{h}(y,\cdot)$ is $\underline{p}^y$-recurrent. □

### The Representation Formula

Recall from Definition 7.1.1 the notion of domination. We will work with $\underline{h}$-dominated maps on $\mathcal{M}_0^R$, i.e maps of $\mathrm{Dom}(\mathcal{M}_0^R,\underline{h})$.

**Theorem 7.3.6.** *We have the following bijection*

$$\begin{array}{rcl} \Psi_{\underline{h}} : \mathrm{Dom}(\mathcal{M}_0^R,\underline{h}) & \longrightarrow & \Omega(\mathcal{T}) \\ \psi & \longmapsto & \inf\limits_{y\in\mathcal{M}_0^R}\{\psi(y)+\underline{h}(y,\cdot)\} \end{array} \tag{7.3.15}$$

*with its inverse being the restriction map*

$$\begin{array}{rcl} \Phi_{\underline{h}} : \Omega(\mathcal{T}) & \longrightarrow & \mathrm{Dom}(\mathcal{M}_0^R,\underline{h}) \\ v & \longmapsto & v_{|\mathcal{M}_0^R} \end{array} \tag{7.3.16}$$

*Proof.* Note that the maps of the form $\inf\limits_{y\in\mathcal{M}_0^R}\{\psi(y)+\underline{h}(y,\cdot)\}$ are viscosity solutions due to Proposition 6.2.1. They are bounded and globally defined. Hence, they belong to $\Omega(\mathcal{T})$ by Theorem 6.0.2. This justifies the well-definition of the map $\Psi_{\underline{h}}$. We need to prove that $\Phi_{\underline{h}}$ is well-defined. Let $v$ be an element of $\Omega(\mathcal{T})$. We show that $v_{|\mathcal{M}_0^R}$ is $\underline{h}$-dominated. Let $x$ and $y$ be two elements of $\mathcal{M}_0^R$ and $\tilde{x}$ the lift of $x$ to $\tilde{\mathcal{M}}_0^R$. Consider the curve $x(t) = \pi\circ\phi_L^t(\tilde{x})$. We know from Proposition 7.2.1 that $x(t)$ is calibrated by $v$ and by any weak-KAM solution

$u$. Then for all negative time $t$ we have

$$v(x) - v(t, x(t)) = h^{t,0}(x(t), x) = u(x) - u(t, x(t))$$

Since $x(-p_n^x)$ converges to $x$, we have for $t = -p_n^x$

$$\lim_n v(x) - v(-p_n^x, x(-p_n^x)) = \lim_n u(x) - u(-p_n^x, x(-p_n^x)) = \lim_n u(x) - u(x(-p_n^x)) = 0$$

We now use the definition of the Lax-Oleinik operator $\mathcal{T}$ to deduce that

$$v(y) - v(x) = \lim_n v(0, y) - v(-p_n^x, x(-p_n^x)) \leq \liminf_n h^{p_n^x}(x(-p_n^x), y) = \liminf_n h^{p_n^x}(x, y) = h^{\underline{p}^x}(x, y) = \underline{h}(x, y)$$

where we used the liminf property (7.3.5) of the Barrier $h^{\underline{p}^x}$. We obtained the desired $\underline{h}$-domination which justifies the well-definition of the map $\Phi_{\underline{h}}$.

Let us show that the map $\Phi_{\underline{h}}$ is a left inverse of the map $\Psi_{\underline{h}}$. Take $v = \Psi_{\underline{h}}(\psi) = \inf_{y \in \mathcal{M}_0^R} \{\psi(y) + \underline{h}(y, \cdot)\} \in \Omega(\mathcal{T})$. Then, for all $x$ and $y$ in $\mathcal{M}_0^R$, the domination condition gives the inequality

$$\psi(x) + \underline{h}(x, x) = \psi(x) \leq \psi(y) + \underline{h}(y, x) \tag{7.3.17}$$

where we recall from Property 3 of Proposition 7.3.5 that $\underline{h}(x, x) = h^{\underline{p}^x}(x, x) = 0$. We obtain for all $x \in \mathcal{M}_0^R$

$$v(x) = \Psi_{\underline{h}}(\psi)(x) = \inf_{y \in \mathcal{M}_0^R} \{\psi(y) + \underline{h}(y, x)\} = \psi(x) + \underline{h}(x, x) = \psi(x)$$

In other words, $\Phi_{\underline{h}} \circ \Psi_{\underline{h}}(\psi) = \psi$.

We now show that $\Phi_{\underline{h}}$ is the right inverse of the map $\Psi_{\underline{h}}$. Let $v \in \Omega(\mathcal{T})$ and consider $w = \Psi_{\underline{h}} \circ \Phi_{\underline{h}}(v) \in \Omega(\mathcal{T})$. We need to prove the these two maps $v$ and $w$ are equal. By the uniqueness Theorem 7.2.4, it suffices to prove that they coincide on the Mather set $\mathcal{M}_0$. Let $x$ be an element of $\mathcal{M}_0^R$. We infer from the $\underline{h}$-domination of $v$ on $\mathcal{M}_0^R$ and from (7.3.17) that

$$w(x) = \Psi_{\underline{h}}(v_{|\mathcal{M}_0^R})(x) = \inf_{y \in \mathcal{M}_0^R} \{v(y) + \underline{h}(y, x)\} = v(x) + \underline{h}(x, x) = v(x)$$

Thus, $w_{|\mathcal{M}_0^R} = v_{|\mathcal{M}_0^R}$ and by continuity of viscosity solutions and density of $\mathcal{M}_0^R$ in $\mathcal{M}_0$, we deduce that $w_{|\mathcal{M}_0} = v_{|\mathcal{M}_0}$. We have shown that $v$ and $w$ are two elements of $\Omega(\mathcal{T})$ which coincide on the Mather set $\mathcal{M}_0$, then we get from the uniqueness Theorem 7.2.4 the equality $v = w = \Psi_{\underline{h}} \circ \Phi_{\underline{h}}(v)$. This concludes the proof of the theorem. □

**Remark 7.3.7.** 1. As stated in the beginning of this subsection, the proof shows that

$\mathcal{M}_0^R$ can be replaced by any of its dense subsets.

2. Note that for all $\underline{h}$-dominated map $\psi \in \mathrm{Dom}(\mathcal{M}_0^R, \underline{h})$, we have $\psi = \Phi_{\underline{h}} \circ \Psi_{\underline{h}}(\psi) = \Psi_{\underline{h}}(\psi)_{|\mathcal{M}_0^R}$ where $\Psi_{\underline{h}}(\psi) \in \Omega(\mathcal{T})$ is continuous. This implies that all elements $\psi$ of $\mathrm{Dom}(\mathcal{M}_0^R, \underline{h})$ are continuous. The reason of that will be explained by the fact that the domination by the barrier $\underline{h}$ implies the domination by another continuous barrier $\underline{k}$ introduced in the next subsection.

Following the last remark, it is possible to restrict the set of points $y$ over which we take the infimum to a subset of $\mathcal{M}_0$, which may not be dense. This reduction is carried out in the next subsection.

### 7.3.2 Representation Formula on $\mathcal{M}_0$

This subsection introduces a new generalized barrier $\underline{k}$, defined on the entire Mather set $\mathcal{M}_0$. This barrier is independent of the choice of sequences $\underline{p}^x$ and satisfies the triangular inequality. These properties make $\underline{k}$ suitable for the more general representation formula presented in Theorem 7.1.2, which we prove in this subsection.

#### The Generalized Peierls Barrier $\underline{k}$

We begin by introducing a generalized Peierls barrier $\underline{k}$, derived from the barrier $\underline{h}$, by enforcing the triangular inequality. This, as we will see, ensures the continuity of $\underline{k}$ and results in a canonical barrier that is independent of the choice of sequences $\underline{p}^x$ (Corollary 7.3.16).

**Definition 7.3.8.** 1. We define the *generalized Peierls Barrier* $\underline{k} : \mathcal{M}_0^R \times M \to \mathbb{R}$ by

$$\underline{k}(x,y) = \inf \left\{ \sum_{i=0}^{N-1} \underline{h}(x_i, x_{i+1}) \mid x_0 = x,\ x_N = y,\ x_i \in \mathcal{M}_0^R,\ N \geq 1 \right\} \tag{7.3.18}$$

with time dependence

$$\underline{k}(t,x,y) = \underline{k}^t(x,y) = \inf \left\{ \sum_{i=0}^{N-2} \underline{h}(x_i, x_{i+1}) + \underline{h}^t(x_{N-1}, y) \mid x_0 = x,\ x_N = y,\ x_i \in \mathcal{M}_0^R,\ N \geq 1 \right\} \tag{7.3.19}$$

2. We define the map $\underline{d} : \mathcal{M}_0^R \times \mathcal{M}_0^R \to \mathbb{R}$ by

$$\underline{d}(x,y) = \underline{k}(x,y) + \underline{k}(y,x) \tag{7.3.20}$$

**Proposition 7.3.9.** *We have*

1. *(Viscosity Solution) For all $x \in \mathcal{M}_0^R$, the map $\underline{k}(\cdot, x, \cdot)$ is a viscosity solution of the Hamilton-Jacobi equation* (5.0.1).

2. *For all* $(t,x,y)\in\mathbb{R}\times\mathcal{M}_0^R\times M$*, we have*

$$h^{\infty+t}(x,y)\le \underline{k}^t(x,y)\le \underline{h}^t(x,y) \tag{7.3.21}$$

3. *For all* $x\in\mathcal{M}_0^R$*,* $\underline{k}(x,x)=0$*.*
4. *(Triagular Inequality) For all times* $t\in\mathbb{R}$ *and all points* $x,y\in\mathcal{M}_0^R$ *and* $z\in M$*, we have the triangular inequality*

$$\underline{k}^t(x,z)\le \underline{k}(x,y)+\underline{k}^t(y,z) \tag{7.3.22}$$

5. *(Regularity) The barrier* $\underline{k}^t$ *is* $\kappa_\varepsilon$*-Lipschitz on* $\mathcal{M}_0^R\times M$ *for all* $\varepsilon>0$ *with* $\kappa_\varepsilon$ *being the Lipschitz constant introduced in Proposition 5.1.13.*

*Proof.* 1. Let $x$ be a point of $\mathcal{M}_0^R$. We have

$$\underline{k}^t(x,y)=\inf\left\{\sum_{i=0}^{N-2}\underline{h}(x_i,x_{i+1})+\underline{h}^t(x_{N-1},y)\mid x_0=x,\ x_N=y,\ x_i\in\mathcal{M}_0^R,\ N\ge 1\right\}$$

By Proposition 7.3.5, this infimum is taken over viscosity solutions. Hence, we infer from Proposition 6.2.1 that $\underline{k}(\cdot,x,\cdot)$ is also a viscosity solution. Moreover, following the regularity Property 2 of Proposition 7.3.5, we deduce that this solution is $\kappa_\varepsilon$-Lipschitz for all $\varepsilon>0$.

2. The second inequality follows immediately from the definition of $\underline{k}$. We prove the first inequality. Let $(x,y)$ be an element of $\mathcal{M}_0^R\times M$ and let $(x_i)_{0\le x_i\le N-1}$ be a sequence of elements of $\mathcal{M}_0^R$ with $x_0=x$ and set $x_N=y$. we have

$$\begin{aligned}
\sum_{i=0}^{N-2}\underline{h}(x_i,x_{i+1})+\underline{h}^t(x_{N-1},y)&=\sum_{i=0}^{N-2}h^{\underline{p}^{x_i}}(x_i,x_{i+1})+h^{\underline{p}^{x_{N-1}}+t}(x_{N-1},y)\\
&=\sum_{i=0}^{N-2}\liminf_{n_i}h^{p^{x_i}_{n_i}}(x_i,x_{i+1})+\liminf_{n_{N-1}}h^{p^{x_{N-1}}_{n_{N-1}}+t}(x_{N-1},y)\\
&=\liminf_{n_1,..,n_{N-1}}\sum_{i=0}^{N-2}h^{p^{x_i}_{n_i}}(x_i,x_{i+1})+h^{p^{x_{N-1}}_{n_{N-1}}+t}(x_{N-1},y)\\
&\ge\liminf_{n_1,..,n_{N-1}}h^{\sum_{i=0}^{N-1}p^{x_i}_{n_i}+t}(x,y)\\
&\ge\liminf_{n}h^{n+t}(x,y)=h^{\infty+t}(x,y)
\end{aligned}$$

Taking the infimum on such sequences, we deduce the inequality $\underline{k}\ge h^\infty$.

3. Let $x$ be a point of $\mathcal{M}_0^R$. By the previous property, we have

$$h^\infty(x,x)\le\underline{k}(x,x)\le\underline{h}(x,x)$$

Moreover, we know from Proposition 5.2.16 and Property 3 of Proposition 7.3.5 that $h^\infty(x,x) = \underline{h}(x,x) = 0$. Hence, $\underline{k}(x,x) = 0$.

4. Let $x$ and $y$ be points of $\mathcal{M}_0^R$ and let $z$ be a point of $M$. Consider two sequences of points $(x_i)_{0\le i\le N}$ with $(x_0, x_N) = (x,y)$ and $(y_j)_{0\le j\le N'}$ with $(y_0, y'_N) = (y,z)$. Note that $x_N = y_0 = y$. Hence, concatenating them into a third sequence $(z_i)_{0\le k\le N+N'}$ and using the definition of the barrier $\underline{k}$, we obtain

$$\underline{k}(x,z) \le \sum_{i=0}^{N+N'-2} \underline{h}(z_i, z_{i+1}) + \underline{h}^t(z_{N+N'-1}, z_{N+N'}) = \sum_{i=0}^{N-1} \underline{h}(x_i, x_{i+1}) + \sum_{j=0}^{N'-2} \underline{h}(y_j, y_{j+1}) + \underline{h}^t(y_{N'-1}, y_{N'})$$

Taking the infimum over such sequences yields the desired triangular inequality.

5. Let $(t,x,y)$ and $(t',x',y')$ be two elements of $\mathbb{R}\times\mathcal{M}_0^R\times M$. We have

$$|\underline{k}^t(x,y) - \underline{k}^{t'}(x',y')| \le |\underline{k}^t(x,y) - \underline{k}^t(x',y)| + |\underline{k}^t(x',y) - \underline{k}^{t'}(x',y')| \tag{7.3.23}$$

We have seen in the proof of the first point that

$$|\underline{k}^t(x',y) - \underline{k}^{t'}(x',y')| \le \kappa_\varepsilon.[d(y,y') + |t-t'|] \tag{7.3.24}$$

We need to bound the other term of the rand-hand side of (7.3.23). We know from the triangular inequality that

$$\underline{k}^t(x,y) \le \underline{k}(x,x') + \underline{k}^t(x',y) \quad \text{and} \quad \underline{k}^t(x',y) \le \underline{k}(x',x) + \underline{k}^t(x,y) \tag{7.3.25}$$

Thus, we get

$$|\underline{k}^t(x,y) - \underline{k}^t(x',y)| \le \max\left\{|\underline{k}(x,x')|, |\underline{k}(x',x)|\right\}$$

Moreover, we proved that $\underline{k}(x,x) = \underline{k}(x',x') = 0$, which yields

$$|\underline{k}(x,x')| = |\underline{k}(x,x') - \underline{k}(x,x)| \le \kappa_\varepsilon.d(x,x')$$

and similarly $|\underline{k}(x',x)| \le \kappa_\varepsilon.d(x,x')$. This leads to the bounding

$$|\underline{k}^t(x,y) - \underline{k}^t(x',y)| \le \kappa_\varepsilon.d(x,x') \tag{7.3.26}$$

Gathering (7.3.23),(7.3.24) and (7.3.26), we conclude that

$$|\underline{k}^t(x,y) - \underline{k}^{t'}(x',y')| \le \kappa_\varepsilon.[d(x,x') + d(y,y') + |t-t'|]$$

which is the desired Lipschitz inequality. $\square$

**Corollary 7.3.10.** *For all time $t \in \mathbb{R}$, the barrier $\underline{k}^t$ extends in a unique way to the set $\mathcal{M}_0 \times M$. The extended barrier $\underline{k} : \mathbb{R} \times \mathcal{M}_0 \times M \to \mathbb{R}$ possesses all the properties featured in Proposition 7.3.9.*

*Proof.* The Property 5 of Proposition 7.3.9 implies that the map $\underline{k}^t$ is uniformly continuous on the dense subset $\mathcal{M}_0^R \times M$ of the compact set $\mathcal{M}_0 \times M$. Hence, it extends uniquely to $\mathcal{M}_0 \times M$.

All the properties extensions are straightforward except for the viscosity solutions. We prove that for all $x \in \mathcal{M}_0$, the map $\underline{k}(\cdot, x, \cdot)$ is a viscosity solution of the Hamilton-Jacobi equation (5.1.8). Let $x$ be in $\mathcal{M}_0$ and let $x_n$ be a sequence of $\mathcal{M}_0^R$ that converges to $x$. For all times $s < t$, the non-expensiveness of the Lax-Oleinik semi-group stated in Proposition 6.1.1 leads to

$$\|\mathcal{T}^{s,t}\underline{k}^s(x_n, \cdot) - \mathcal{T}^{s,t}\underline{k}^s(x, \cdot)\|_\infty \leq \|\underline{k}^s(x_n, \cdot) - \underline{k}^s(x, \cdot)\|_\infty \leq \kappa_1.d(x_n, x) \longrightarrow 0 \quad \text{as } n \to +\infty$$

Then $\lim_n \mathcal{T}^{s,t}\underline{k}^s(x_n, \cdot) = \mathcal{T}^{s,t}\underline{k}^s(x, \cdot)$.

Moreover, since $\underline{k}(\cdot, x_n, \cdot)$ is a viscosity solution, we have $\mathcal{T}^{s,t}\underline{k}^s(x_n, \cdot) = \underline{k}^t(x_n, \cdot)$ with

$$\|\underline{k}^t(x_n, \cdot) - \underline{k}^t(x, \cdot)\|_\infty \leq \kappa_1.d(x_n, x) \longrightarrow 0 \quad \text{as } n \to +\infty$$

Therefore

$$\mathcal{T}^{s,t}\underline{k}^s(x, \cdot) = \mathcal{T}^{s,t}\underline{k}^s(x_n, \cdot) = \lim_n \underline{k}^t(x_n, \cdot) = \underline{k}^t(x, \cdot)$$

and $\underline{k}(\cdot, x, \cdot)$ is a viscosity solution. $\square$

### The Generalized Representation Formula

Before diving into the proof of the main result of this chapter, let us give some representation formulas that follow directly from Theorem 7.3.6.

**Proposition 7.3.11.** *1. We have equality $\mathrm{Dom}(\mathcal{M}_0^R, \underline{k}) = \mathrm{Dom}(\mathcal{M}_0^R, \underline{h})$.*

*2. The bijection $\Psi_{\underline{h}}$ expressed in (7.3.15) is equal to*

$$\begin{array}{ccc} \mathrm{Dom}(\mathcal{M}_0^R, \underline{k}) & \longrightarrow & \Omega(\mathcal{T}) \\ \psi & \longmapsto & \inf_{y \in \mathcal{M}_0^R} \{\psi(y) + \underline{k}(y, \cdot)\} \end{array} \tag{7.3.27}$$

*3. Extending this formula by continuity to the Mather set $\mathcal{M}_0$, we get the bijection*

$$\begin{array}{ccc} \mathrm{Dom}(\mathcal{M}_0, \underline{k}) & \longrightarrow & \Omega(\mathcal{T}) \\ \psi & \longmapsto & \inf_{y \in \mathcal{M}_0} \{\psi(y) + \underline{k}(y, \cdot)\} \end{array} \tag{7.3.28}$$

**Remark 7.3.12.** We can note that the equality $\mathrm{Dom}(\mathcal{M}_0^R, \underline{k}) = \mathrm{Dom}(\mathcal{M}_0^R, \underline{h})$ explains the continuity of $\underline{h}$-dominated maps deduced from Theorem 7.3.6.

*Proof. Domination.* The inclusion $\mathrm{Dom}(\mathcal{M}_0^R, \underline{k}) \subset \mathrm{Dom}(\mathcal{M}_0^R, \underline{h})$ comes from the inequality $\underline{k} \leq \underline{h}$. Let us prove the inverse inclusion. Fix a map $\psi \in \mathrm{Dom}(\mathcal{M}_0^R, \underline{h})$. For all sequence $(x_i)_{0\leq i\leq N}$ in $\mathcal{M}_0^R$, we have

$$\psi(x_{i+1}) - \psi(x_i) \leq \underline{h}(x_i, x_{i+1})$$

Thus, summing on $i$ yields

$$\psi(x_N) - \psi(x_0) \leq \sum_{i=0}^{N-1} \underline{h}(x_i, x_{i+1})$$

Then, taking the infimum on such sequences linking $x_0 = x$ to $x_N = y$ in $\mathcal{M}_0^R$, we obtain the domination inequality

$$\psi(y) - \psi(x) \leq \underline{k}(x, y)$$

which shows that $\psi$ belongs to $\mathrm{Dom}(\mathcal{M}_0^R, \underline{k})$.

*Formula.* Denote by $\widetilde{\Psi}_{\underline{h}}$ the map expressed in (7.3.27). Let $\psi$ be an element of $\mathrm{Dom}(\mathcal{M}_0^R, \underline{k}) = \mathrm{Dom}(\mathcal{M}_0^R, \underline{h})$ and consider $v = \Psi_{\underline{h}}(\psi)$ and $\tilde{v} = \widetilde{\Psi}_{\underline{h}}(\psi)$. As in the proof of Theorem 7.3.6, we have $v_{|\mathcal{M}_0^R} = \tilde{v}_{|\mathcal{M}_0^R} = \psi$. Hence, by density of $\mathcal{M}_0^R$ in the Mather set $\mathcal{M}_0$ and by continuity of the maps $v$ and $\tilde{v}$, we deduce that $v_{|\mathcal{M}_0} = \tilde{v}_{|\mathcal{M}_0}$. Since $v$ and $\tilde{v}$ belong to the non-wandering set $\Omega(\mathcal{T})$, we conclude using the Uniqueness Theorem 7.2.4 that $\Psi_{\underline{h}}(\psi) = v = \tilde{v} = \widetilde{\Psi}_{\underline{h}}(\psi)$. □

We now establish the representation formula for the non-wandering set $\Omega(\mathcal{T})$ using the generalized Peierls barrier $\underline{k}$. To reduce the set over which the infimum is taken, we introduce an equivalence relation defined by the map $\underline{d}$ which is a pseudometric due to the following proposition.

**Proposition 7.3.13.** *The map $\underline{d}$ is a pseudometric on $\mathcal{M}_0$.*

*Proof.* The symmetry $\underline{d}(x, y) = \underline{d}(y, x)$ is immediate. Moreover, we know from the Property 3 of Proposition 7.3.9 that for all $x \in \mathcal{M}_0$, $\underline{d}(x, x) = 2.\underline{k}(x, x) = 0$.

*Non-Negativity.* For any elements $x$ and $y$ of $\mathcal{M}_0$, the Triangular inequality (7.3.22) yields

$$\underline{d}(x, y) = \underline{k}(x, y) + \underline{k}(y, x) \geq \underline{k}(x, x) = 0$$

*Triangular Inequality.* Fix three elements $x$, $y$ and $z$ in $\mathcal{M}_0$. Applying the triangular inequality (7.3.22) twice, we get

$$\begin{aligned}\underline{d}(x,z) &= \underline{k}(x,z) + \underline{k}(z,x)\\ &\leq \underline{k}(x,y) + \underline{k}(y,z) + \underline{k}(z,y) + \underline{k}(y,x)\\ &\leq \underline{d}(x,y) + \underline{d}(y,z)\end{aligned}$$

□

**Definition 7.3.14.** 1. We define the equivalence relation $\sim$ on $\mathcal{M}_0$ by

$$x \sim y \Longleftrightarrow \underline{d}(x,y) = 0 \tag{7.3.29}$$

2. The *generalized static classes* are the equivalence classes of the equivalence relation $\sim$. We denote by $\underline{\mathbb{M}}$ the set of generalized static classes. We represent every element of $\underline{\mathbb{M}}$ by an element of $\mathcal{M}_0$ so that we have the inclusion $\underline{\mathbb{M}} \subset \mathcal{M}_0$.

Recall from Definition 7.1.1 the notion of domination. We will work with the set $\mathrm{Dom}(\underline{\mathbb{M}}, \underline{k})$ of $\underline{k}$-dominated maps on the set of generalized static classes $\underline{\mathbb{M}}$. We can now prove the main result of this chapter.

*Proof of Theorem 7.1.2.* We saw in Proposition 7.3.11 that $\mathrm{Dom}(\mathcal{M}_0^R, \underline{k}) = \mathrm{Dom}(\mathcal{M}_0^R, \underline{h})$. And we know from Theorem 7.3.6 that all the elements of $\Omega(\mathcal{T})$ are $\underline{h}$-dominated on $\mathcal{M}_0^R$. Hence, they are $\underline{k}$-dominated on $\mathcal{M}_0^R$ and by continuity on the Mather set $\mathcal{M}_0$. In particular, $\underline{k}$-domination holds in $\underline{\mathbb{M}} \subset \mathcal{M}_0$. Following the proof of Theorem 7.3.6, this allows to prove that the maps $\Psi_{\underline{k}}$ and $\Phi_{\underline{k}}$ are well-defined and that $\Phi_{\underline{k}} \circ \Psi_{\underline{k}} = Id_{\mathrm{Dom}(\underline{\mathbb{M}}, \underline{k})}$. However, the identity $\Psi_{\underline{k}} \circ \Phi_{\underline{k}} = Id_{\Omega(\mathcal{T})}$ requires to show that the Uniqueness Theorem 7.2.4 holds on the set of generalized static classes $\underline{\mathbb{M}}$.

Let us show this. Let $v \in \Omega(\mathcal{T})$ and consider $w = \Psi_{\underline{k}} \circ \Phi_{\underline{k}}(v) \in \Omega(\mathcal{T})$. We need to prove that $v$ and $w$ are equal. By the uniqueness Theorem 7.2.4, it suffices to prove that they coincide on the Mather set $\mathcal{M}_0$. Let $x$ be an element of $\underline{\mathbb{M}} \subset \mathcal{M}_0$. Then due to the $\underline{k}$-domination of $v_{|\underline{\mathbb{M}}}$, and similarly to (7.3.17), we have

$$w(x) = \Psi_{\underline{k}} \circ \Phi_{\underline{k}}(v)(x) = \inf_{y \in \underline{\mathbb{M}}} \{v(y) + \underline{k}(y,x)\} = v(x) + \underline{k}(x,x) = v(x)$$

Let $x$ be a any element of $\mathcal{M}_0$ and $y \in \underline{\mathbb{M}}$ such that $x \sim y$. Then, due to the $\underline{k}$-domination of $v_{|\mathcal{M}_0}$, we have the inequalities

$$v(x) - v(y) \leq \underline{k}(y,x) \quad \text{and} \quad v(y) - v(x) \leq \underline{k}(x,y)$$

Taking the sum, we obtain

$$0 = [v(x) - v(y)] + [v(y) - v(x)] \leq \underline{k}(y,x) + \underline{k}(x,y) = \underline{d}(y,x) = 0$$

which implies equality in the two inequalities and

$$v(x) - v(y) = \underline{k}(y,x) \quad \text{and} \quad v(y) - v(x) = \underline{k}(x,y)$$

Moreover, for all $z \in \underline{\mathbb{M}}$, the $\underline{k}$-domination writes

$$v(x) \leq v(z) + \underline{k}(z,x)$$

Therefore, we obtain

$$w(x) = \inf_{z \in \underline{\mathbb{M}}} \{v(z) + \underline{k}(z,x)\} = v(y) + \underline{k}(y,x) = v(x) \tag{7.3.30}$$

The maps $v$ and $w$ are two elements of $\Omega(\mathcal{T})$ which coincide on the Mather set $\mathcal{M}_0$. Then the Uniqueness Theorem 7.2.4 results in the equality $v = w = \Psi_{\underline{k}} \circ \Phi_{\underline{k}}(v)$. □

**Remark 7.3.15.** It was noted in Remark 7.3.7 that $\mathcal{M}_0^R$ can be replaced by any dense subset $\mathcal{M}_0'$. It is still possible to consider the generalized static classes $\underline{\mathbb{M}}'$ inside $\mathcal{M}_0'$. Taking countable dense sets allows to avoid the use of the axiom of choice.

We conclude this section by providing an application of this representation formula which shows that the generalized Peierls barrier $\underline{k}$ does not depend on the choice of the sequences $\underline{p}^x$ used to define the barrier $\underline{h}$.

**Corollary 7.3.16.** *For all points $x_0 \in \mathcal{M}_0$ and $x \in M$, we have the formulas*

$$\begin{aligned} \max_{v \in \Omega(\mathcal{T})} \{v(x) - v(x_0)\} &= \underline{k}(x_0, x) \\ \min_{v \in \Omega(\mathcal{T})} \{v(x) - v(x_0)\} &= \inf_{\substack{y \in \underline{\mathbb{M}} \\ y \nsim x_0}} \{-\underline{k}(y, x_0) + \underline{k}(y,x)\} \end{aligned} \tag{7.3.31}$$

*Proof.* Let us prove the first identity. We will use the representation formula (7.3.28). We consider $\psi^+ : \mathcal{M}_0 \to \mathbb{R}$ defined by $\psi^+(x) = \underline{k}(x_0, x)$. We know from the Property 3 of Proposition 7.3.9 that $\psi^+(x_0) = \underline{k}(x_0, x_0) = 0$, and from the triangular inequality (7.3.22) that for all $y \in \mathcal{M}_0$,

$$\underline{k}(x_0, x) \leq \underline{k}(x_0, y) + \underline{k}(y, x)$$

which yields

$$\psi^+(x) - \psi^+(y) = \underline{k}(x_0, x) - \underline{k}(x_0, y) \leq \underline{k}(x, y)$$

Then, $\psi^+$ belongs to $\mathrm{Dom}(\mathcal{M}_0, \underline{k})$. We consider the map $v^+ = \Psi_{\underline{k}}(\psi^+) \in \Omega(\mathcal{T})$. We have

$v^+(x_0) = \psi^+(x_0) = 0$. Hence, the triangular inequality results in

$$v^+(x) = \inf_{y\in\mathcal{M}_0} \{\underline{k}(x_0,y) + \underline{k}(y,x)\} = \underline{k}(x_0,x)$$

We showed that $v^+ = \Psi_{\underline{k}}(\psi^+) = \underline{k}(x_0,\cdot)$.

Additionally, for all $\psi \in \mathrm{Dom}(\mathcal{M}_0,\underline{k})$ such that $\psi(x_0) = 0$, the domination condition yields for all $x \in \mathcal{M}_0$

$$\psi(x) \leq \inf_{y\in\mathcal{M}_0} \{\psi(y) + \underline{k}(y,x)\} \leq \psi(x_0) + \underline{k}(x_0,x) = \underline{k}(x_0,x) = \psi^+(x)$$

Hence, for all $v = \Psi_{\underline{k}}(\psi)$, we have

$$v(x) = \inf_{y\in\mathcal{M}_0} \{\psi(y) + \underline{k}(y,x)\} \leq \inf_{y\in\mathcal{M}_0} \{\psi^+(y) + \underline{k}(y,x)\} = v^+(x) = \underline{k}(x_0,x)$$

Since all elements $v \in \Omega(\mathcal{T})$ with $v(x_0) = 0$ are of the form $\Psi_{\underline{k}}(\psi)$ with $\psi \in \mathrm{Dom}(\mathcal{M}_0,\underline{k})$ and $\psi(x_0) = 0$, we deduce the first identity of (7.3.31).

We now prove the second formula. For this formula, we use the representation formula of Theorem 7.1.2. Consider $\psi^- : \underline{\mathbb{M}} \to \mathbb{R}$ defined by $\psi^-(x) = -\underline{k}(x,x_0)$. We have by triangular inequality that

$$\psi^-(x) - \psi^-(y) = \underline{k}(y,x_0) - \underline{k}(x,x_0) \leq \underline{k}(y,x)$$

Thus, $\psi^-$ belongs to $\mathrm{Dom}(\underline{\mathbb{M}},\underline{k})$. We consider the map $v^- = \Psi_{\underline{k}}(\psi^-) \in \Omega(\mathcal{T})$ and let $x_1$ be in $\underline{\mathbb{M}}$ such that $x_0 \sim x_1$. As proved by identity (7.3.30), we have

$$v^-(x_0) = \psi^-(x_1) + \underline{k}(x_1,x_0) = -\underline{k}(x_1,x_0) + \underline{k}(x_1,x_0) = 0$$

We claim that for all $x \in M$,

$$\underline{k}(x_1,x) = \underline{k}(x_1,x_0) + \underline{k}(x_0,x)$$

Indeed, we apply the triangular inequality (7.3.22) twice to get

$$\begin{aligned}\underline{k}(x_1,x) &\leq \underline{k}(x_1,x_0) + \underline{k}(x_0,x)\\ &\leq \underline{k}(x_1,x_0) + \underline{k}(x_0,x_1) + \underline{k}(x_1,x)\\ &= \underline{d}(x_0,x_1) + \underline{k}(x_1,x) = \underline{k}(x_1,x)\end{aligned}$$

where we used the definition $\underline{d}(x_0,x_1) = 0$ of $x_0 \sim x_1$. Hence, we deduce the equality everywhere and the claim is proved.

Consequently, we get for all $y \in \underline{\mathbb{M}}$

$$\psi^-(y) + \underline{k}(y,x) = -\underline{k}(y,x_0) + \underline{k}(y,x) \le \underline{k}(x_0,x) = -\underline{k}(x_1,x_0) + \underline{k}(x_1,x) = \psi^-(x_1) + \underline{k}(x_1,x)$$

and

$$v^-(x) = \inf_{y \in \underline{\mathbb{M}}} \{\psi^-(y) + \underline{k}(y,x)\} = \inf_{\substack{y \in \underline{\mathbb{M}} \\ y \nsim x_0}} \{-\underline{k}(y,x_0) + \underline{k}(y,x)\}$$

Let $\psi$ be an element of $\mathrm{Dom}(\underline{\mathbb{M}}, \underline{k})$ such that $\psi(x_0) = 0$ and $v = \Psi_{\underline{k}}(\psi)$. The domination condition yields

$$\psi(x) \ge \sup_{y \in \underline{\mathbb{M}}} \{\psi(y) - \underline{k}(x,y)\} \ge \psi(x_0) - \underline{k}(x,x_0) = \psi^-(x)$$

Therefore, $v \ge v^-$ and we obtain the desired identity. □

## 7.4 Applications

We explore several applications of the representation formula for different examples of Tonelli Hamiltonians.

First, we examine autonomous Tonelli Hamiltonians, for which we will prove Fathi's Convergence Theorem 7.1.3. Next, we will treat the case of Tonelli Hamiltonians with $N$-periodic or $\underline{p}$-recurrent Mather sets.

Additionally, the representation formula can be used to identify specific subsets of the non-wandering set $\Omega(\mathcal{T})$, such as $\mathrm{Fix}(\mathcal{T})$ and $\mathrm{Per}_n(\mathcal{T})$. The representation of weak-KAM solutions of $\mathrm{Fix}(\mathcal{T})$ in the non-autonomous case was established by G. Contreras, R. Iturriaga, and H. Sánchez-Morgado in [CISM13].

It should be noted that these applications are not direct results of the theorems but rather adaptations of their proofs.

### 7.4.1 The Autonomous Case

We consider the Autonomous framework. Let $H : T^*M \to \mathbb{R}$ be a Tonelli Hamiltonian with corresponding Lagrangian $L : TM \to \mathbb{R}$. In this case, the studied objects $\phi_L^{s,t}$, $\mathcal{T}^{s,t}$, $h^{s,t}$,etc.. verify $\phi_L^{s,t} = \phi_L^{t-s}$, $\mathcal{T}^{s,t} = \mathcal{T}^{t-s}$, $h^{s,t} = h^{t-s}$,etc..

Note that the minimizing measures used to define the Mather set in (5.2.2) are defined on $TM$ instead of $\mathbb{T}^1 \times TM$ so that the Mather set $\tilde{\mathcal{M}} = \tilde{\mathcal{M}}_0$ is included in $TM$.

Let us first prove the following weaker version of Fathi's Theorem 7.1.3 following ideas of P.Bernard and J-M.Roquejoffre used in [BR04].

**Theorem 7.4.1.** *For an autonomous Tonelli Hamiltonian $H : T^*M \to \mathbb{R}$, we have $\Omega(\mathcal{T}) = \mathrm{Fix}(\mathcal{T})$ and the representation formula is given by the following bijection*

$$\begin{array}{cccc} \Psi_1 : \mathrm{Dom}(\underline{\mathbb{M}}, h^\infty) & \longrightarrow & \Omega(\mathcal{T}) \\ \psi & \longmapsto & \inf\limits_{y \in \underline{\mathbb{M}}} \{\psi(y) + h^\infty(y, \cdot)\} \end{array} \tag{7.4.1}$$

*with its inverse being the restriction map*

$$\begin{array}{cccc} \Phi_1 : \Omega(\mathcal{T}) & \longrightarrow & \mathrm{Dom}(\underline{\mathbb{M}}, h^\infty) \\ v & \longmapsto & v_{|\underline{\mathbb{M}}} \end{array} \tag{7.4.2}$$

*Proof.* Let $v$ in $\Omega(\mathcal{T})$. Fix a point $x_0 \in \mathcal{M}$ with lift $\tilde{x}_0$ in $\tilde{\mathcal{M}}$, set $x(t) = \pi \circ \phi_L^t(\tilde{x})$ and consider the weak-KAM solution $u(x) = h^\infty(x_0, x)$. We will show that the map $f(t,x) = v(t,x) - u(t,x)$ is constant on $\mathbb{R} \times \{x_0(\tau); \tau \in \mathbb{R}\} \subset \mathbb{R} \times \mathcal{M}$. We know from Proposition 7.2.1 that the curve $x_0(t)$ is calibrated by both $u$ and $v$, and we have

$$u(t, x_0(t)) - u(0, x_0) = h^t(x_0, x_0(t)) = v(t, x_0(t)) - v(0, x_0)$$

Hence, we get

$$f(t, x_0(t)) = v(t, x_0(t)) - u(t, x_0(t)) = v(0, x_0) - u(0, x_0) = f(0, x_0)$$

and $f(t, x_0(t))$ is constant on time. Moreover, we know from the regularity on calibrated curves Proposition 5.1.20 that for all $y \in \mathcal{M}$ with lift $\tilde{y} \in \tilde{\mathcal{M}}$, we have

$$d_x f(t,y) = d_x v(t,y) - d_x u(t,y) = \partial_v L(\tilde{y}) - \partial_v L(\tilde{y}) = 0$$

Thus, since the curve $x_0(t)$ is $C^1$ regular, we can integrate along it and get for any time $s \in \mathbb{R}$

$$f(t, x_0(s)) = f(t, x_0) + \int_0^s d_x f(t, x_0(\tau)).\dot{x}_0(\tau)\, d\tau = f(t, x_0)$$

and we deduce that that map $f(t,x)$ is constant on the set $\mathbb{R} \times \{x_0(\tau); \tau \in \mathbb{R}\}$. In particular, using the fact that $h^{\infty - n}(x_0, x_0) = h^\infty(x_0, x_0) = 0$ established in Proposition 5.2.16, we obtain

$$\begin{aligned} v(-n, x_0) &= v(-n, x_0) - u(-n, x_0) = f(-n, x_0) = f(-n, x_0(-n)) \\ &= f(0, x_0) = v(0, x_0) - u(0, x_0) = v(0, x_0) = v(x_0) \end{aligned}$$

which stand for any $x_0 \in \mathcal{M}$. Consequently, and by definition of viscosity solutions, we get for all $(x, y)$ in $\mathcal{M} \times M$

$$v(y) - v(x) = v(y) - v(-n, x) \leq h^n(x, y)$$

and taking the liminf on $n$, we obtain the domination

$$v(y) - v(x) \leq h^\infty(x, y)$$

The identity (7.3.31) implies that for any $(x, y)$ in $\mathcal{M} \times M$,

$$\underline{k}(x, y) = \max_{v \in \Omega(\mathcal{T})} \{v(y) - v(x)\} \leq h^\infty(x, y)$$

and the Property 2 of Proposition 7.3.9 gives the inverse inequality, which results in the equality $\underline{k} = h^\infty$. Therefore, the bijection established in Theorem 7.1.2 translates into (7.4.1). And for all $v = \Psi_1(\psi)$ for some $\psi \in \mathrm{Dom}(\underline{\mathbb{M}}, h^\infty)$, Proposition 6.2.1 gives

$$\begin{aligned}\mathcal{T}v(x) = v(1, x) &= \inf_{y \in \underline{\mathbb{M}}} \{\psi(y) + h^{\infty+1}(y, \cdot)\} \\ &= \inf_{y \in \underline{\mathbb{M}}} \{\psi(y) + h^\infty(y, \cdot)\} = v(x)\end{aligned}$$

which justifies that $v$ belongs to $\mathrm{Fix}(\mathcal{T})$ and hence that $\Omega(\mathcal{T}) = \mathrm{Fix}(\mathcal{T})$. □

**Remark 7.4.2.** 1. Note from the proof that the key point to obtain weak-KAM solutions is to establish the domination by $h^\infty$. We will see in the next examples that these dominations by different barriers can detect the periodicity of the $\underline{p}$-recurrence of viscosity solutions for some sequence $p$.

2. This result is a weaker version of the theorem of A.Fathi [Fat98], which more generally shows that for all times $t$, $\underline{k}^t(x, , y) = h^\infty(x, y) = \lim_{s \to +\infty} h^s(x, y)$. This indicates that the weak-KAM solutions $v \in \Omega(\mathcal{T})$ are independent of the time $t$ and that they are viscosity solutions of the stationary Hamilton-Jacobi equation

$$H(x, d_x u) = \alpha_0 \tag{7.4.3}$$

*Proof of Corollary 7.1.3.* Fix a time $\tau > 0$ and let $\tilde{H}(t, x, p) = H(t + \tau, x, p)$. We add a tilde in the notation of all the objects $\tilde{L}$, $\tilde{\mathcal{T}}$, $\tilde{\alpha}_0$ associated to $\tilde{H}$.

The potential $\tilde{h}_0$ associated to $\tilde{H}$ verifies for all points $x$ and $y$ in $M$ and all times $s < t$

$$\tilde{h}_0^{s,t}(x, y) = \inf\left\{ \int_s^t L(\zeta + \tau, \tilde{\gamma}(\zeta), \dot{\tilde{\gamma}}(\zeta))\, d\zeta \;\middle|\; \begin{array}{rl}\tilde{\gamma}: & [s,t] \to M \\ & s \mapsto x \\ & t \mapsto y\end{array} \right\}$$

$$
\begin{aligned}
&= \inf\left\{ \int_{s+\tau}^{t+\tau} L(\zeta,\gamma(\zeta),\dot\gamma(\zeta))\,d\zeta \;\middle|\; \begin{array}{rl} \gamma: & [s+\tau,t+\tau]\to M \\ & s\mapsto x \\ & t\mapsto y \end{array} \right\} \\
&= h_0^{s+\tau,t+\tau}(x,y)
\end{aligned}
$$

and in particular, its Lax-Oleinik operator $\tilde{\mathcal{T}}_0^t$ verifies $\tilde{\mathcal{T}}_0^t = \mathcal{T}_0^{\tau,t+\tau}$.

Moreover, using the characterization of the Mañé critical value mentioned in Remark 6.1.3, we deduce that $\tilde\alpha_0 = \alpha_0$. Hence, the Peierls barriers verify $\tilde h^\infty = h^{\tau,\infty+\tau}$, the full Lax-Oleinik operator verifies $\tilde{\mathcal{T}}^t = \mathcal{T}^{\tau,t+\tau}$, and the non-wandering set of $\tilde{\mathcal{T}}$ is given by $\Omega(\tilde{\mathcal{T}}) = \Omega(\mathcal{T}^{\tau,1+\tau})$.

Recall from Proposition 6.0.1 the definition of $\Omega_\tau(\mathcal{T}) = \mathcal{T}^\tau\Omega(\mathcal{T})$. We claim that $\Omega(\tilde{\mathcal{T}}) = \Omega_\tau(\mathcal{T})$. Indeed, since by Proposition 6.0.1 the map $\mathcal{T}^\tau$ is invertible in the two sets, we get

$$
\begin{aligned}
\Omega_\tau(\mathcal{T}) = \mathcal{T}^\tau\Omega(\mathcal{T}) &= \{\mathcal{T}^\tau u \in \mathcal{C}(M,\mathbb{R}) \mid u \text{ is a limit point of } \mathcal{T}^n u = \mathcal{T}^{\tau,n}\mathcal{T}^\tau u\} \\
&= \{\mathcal{T}^\tau u \in \mathcal{C}(M,\mathbb{R}) \mid \mathcal{T}^\tau u \text{ is a limit point of } \mathcal{T}^\tau\mathcal{T}^{\tau,n}\mathcal{T}^\tau u = \mathcal{T}^{\tau,n+\tau}\mathcal{T}^\tau u\} \\
&= \{v \in \mathcal{C}(M,\mathbb{R}) \mid u \text{ is a limit point of } \mathcal{T}^{\tau,n+\tau} v\} \\
&= \Omega(\mathcal{T}^{\tau,1+\tau}) = \Omega(\tilde{\mathcal{T}})
\end{aligned}
$$

Therefore, applying the representation formula 7.4.1 once for $\Omega(\tilde{\mathcal{T}})$ on $\mathcal{M}_0(\tilde L)$ and once for $\Omega_\tau(\mathcal{T})$ on $\mathcal{M}_0$, we get the following formulas

$$
\begin{array}{rcl}
\tilde\Psi_1 : \mathrm{Dom}(\mathcal{M}_0(\tilde L), h^{\tau,\infty+\tau}) & \longrightarrow & \Omega_\tau(\mathcal{T}) \\
\psi & \longmapsto & \inf\limits_{y\in\mathcal{M}_0(\tilde L)} \{\psi(y) + h^{\tau,\infty+\tau}(y,\cdot)\}
\end{array}
\tag{7.4.4}
$$

and

$$
\begin{array}{rcl}
\Psi_1^\tau : \mathrm{Dom}(\mathcal{M}_0, h^{\infty}) & \longrightarrow & \Omega_\tau(\mathcal{T}) \\
\psi & \longmapsto & \inf\limits_{y\in\mathcal{M}_0} \{\psi(y) + h^{\infty+\tau}(y,\cdot)\}
\end{array}
\tag{7.4.5}
$$

Since we are working in the autonomous case, we have $\tilde H = H$ and $\mathcal{M}_0(\tilde L) = \mathcal{M}_0$. Hence, Corollary 7.3.16 yields for all $x$ in $\mathcal{M}_0$ and $y$ in $M$

$$
h^\infty(x,y) = h^{\tau,\infty+\tau}(x,y) = \sup_{v\in\Omega_\tau(\mathcal{T})} \{v(y) - v(x)\} = h^{\infty+\tau}(x,y) \tag{7.4.6}
$$

We deduce that all the elements of $\Omega(\mathcal{T})$ are constant in time and hence are solutions of the stationary Hamilton-Jacobi equation (7.4.3). In particular, $\Omega(\mathcal{T}) = \Omega_\tau(\mathcal{T})$. And Theorem 7.4.1asserts that $\Omega(\mathcal{T}) = \mathrm{Fix}(\mathcal{T})$, we obtain that $\mathrm{Fix}(\mathcal{T}) = \bigcap_{t>0}\mathrm{Fix}(\mathcal{T}^t)$.

Now let $u$ be a scalar map in $\mathcal{C}(M,\mathbb{R})$. We have from Proposition 6.3.3 that $\omega(u) \subset$

$\Omega(\mathcal{T}) = \mathrm{Fix}(\mathcal{T})$, and by the minimality property, we deduce that $\omega(u)$ is a singleton so that $\mathcal{T}^n u$ converges to a weak-KAM solution $v$. Hence, for all time $t > 0$, $\mathcal{T}^{n+t}u$ converges to $v(t,\cdot) = v$. Therefore, we deduce that $\mathcal{T}^n u$ converges to the weak-KAM solution $v$ of the Hamilton-jacobi equation (7.4.3). □

### 7.4.2 Periodic Viscosity Solutions

In this subsection, we present a representation formula for periodic viscosity solutions for a fixed integer period $n \geq 1$. This formula generalizes the representation formula for weak-KAM solutions i.e one-time periodic solutions, as established in [CISM13].

To derive such a formula, we need to identify the appropriate domination barrier. This is provided by the $n$-barrier $h^{n\infty}$, which was first introduced by A. Fathi and J. N. Mather in [FM00].

**Definition 7.4.3.** 1. We define the *$n$-Peierls Barrier* $h^{n\infty} : M \times M \to \mathbb{R}$ as the $\underline{p}$-Peierls barrier for $p_k = n.k$, i.e

$$h^{n\infty}(x,y) = \liminf_{p\to\infty} h^{np}(x,y) \tag{7.4.7}$$

2. We define the subset $\mathcal{M}_n$ of the Mather set $\mathcal{M}_0$ by

$$\mathcal{M}_n := \{x \in \mathcal{M}_0 \mid h^{n\infty}(x,x) = 0\} \tag{7.4.8}$$

3. We define the map $d_n : \mathcal{M}_n \times \mathcal{M}_n \to \mathbb{R}$ by

$$d_n(x,y) = h^{n\infty}(x,y) + h^{n\infty}(y,x) \tag{7.4.9}$$

4. We define the equivalence relation $\sim_n$ on $\mathcal{M}_n$ by

$$x \sim_n y \iff d_n(x,y) = 0 \tag{7.4.10}$$

5. The *$n$-static classes* are the equivalence classes of the equivalence relation $\sim_n$. We denote by $\mathbb{M}_n$ the set of $n$-static classes. We assume that every class of $\mathbb{M}_n$ is represented by an element of $\mathcal{M}_n$ so that we have the inclusion $\mathbb{M}_n \subset \mathcal{M}_n \subset \mathcal{M}_0$.

6. We say that a map $\psi : X \to \mathbb{R}$ is $n$-dominated on a set $X$ if it is $h^{n\infty}$-dominated on $X$.

**Remark 7.4.4.** Analogously to Proposition 7.3.13, and due to the triangular inequality (5.2.28) satisfied by the $n$-Peierls barrier $h^{n\infty}$, the map $d_n$ is a pseudometric on $\mathcal{M}_0$, which justifies that $\sim_n$ is an equivalence relation.

We will work on the set $\mathrm{Dom}(\mathbb{M}_n, h^{n\infty})$ of $n$-dominated maps on the set of $n$-static classes $\mathbb{M}_n$ to prove Theorem 7.1.4.

*Proof of Theorem 7.1.4.* We first show that the maps $\Psi_n$ and $\Phi_n$ are well defined. Let $\psi$ be an element of $\mathrm{Dom}(\mathbb{M}_n, h^{n\infty})$ and set $v = \Psi_n(\psi)$. We know from Property 3 of Proposition 7.3.2 that $h^{n\infty}(\cdot, y, \cdot)$ is a viscosity solution for all $y \in \mathbb{M}_n$. Hence, we infer from Proposition 6.2.1 that $v$ is a viscosity solution and that

$$\begin{aligned}\mathcal{T}^n v(x) &= \inf_{y\in\mathbb{M}_n} \{\psi(y) + \mathcal{T}^n h^{n\infty}(y,\cdot)(x)\} \\ &= \inf_{y\in\mathbb{M}_n} \{\psi(y) + h^{n\infty+n}(y,x)\} \\ &= \inf_{y\in\mathbb{M}_n} \{\psi(y) + h^{n\infty}(y,x)\} = v(x)\end{aligned}$$

Thus, $v \in \mathrm{Per}_n(\mathcal{T})$ and $\Psi_n$ is well defined.

Now let $v$ be an element of $\mathrm{Per}_n(\mathcal{T})$ and $x$ and $y$ be two points of $\mathbb{M}_n$. We have by definition of $v$ that for any integer $k$,

$$v(y) - v(x) = v(nk, y) - v(x) \le h^{nk}(x, y)$$

and taking the liminf on $k$ leads to

$$v(y) - v(x) \le h^{n\infty}(x, y)$$

We showed that $\Phi_n(v) = v_{|\mathbb{M}_n}$ is $n$-dominated, which justifies the well-definition of the map $\Phi_n$.

The proof of the identity $\Psi_n \circ \Phi_n = Id_{\mathrm{Dom}(\mathbb{M}_n, h^{n\infty})}$ is analogous what was done in the proof of Theorem 7.3.6. We show that $\Phi_n \circ \Psi_n = Id_{\mathrm{Per}_n(\mathcal{T})}$. Let $v$ be an element of $\mathrm{Per}_n(\mathcal{T})$ and set $w = \Phi_n \circ \Psi_n(v) \in \mathrm{Per}_n(\mathcal{T})$. We aim to prove that $w = v$. By the domination condition and following the proof of Theorem 7.1.2, we get successively that $v_{|\mathbb{M}_n} = w_{|\mathbb{M}_n}$ and $v_{|\mathcal{M}_n} = w_{|\mathcal{M}_n}$. Let $x$ be an element of the Mather set $\mathcal{M}_0$ with lift $\tilde{x}$ in $\tilde{\mathcal{M}}_0$ and set $x(t) = \pi \circ \phi_L^t(\tilde{x})$. By compactness of $M$, there exist an increasing sequence $k_i$ of integers such that $\lim_i k_{i+1} - k_i = +\infty$ and $x(-nk_i)$ converges to a point $x_\alpha$ belonging to the closed set $\mathcal{M}_0$. We set $x_\alpha(t) = \pi \circ \phi_L^t(\tilde{x}_\alpha)$ with $\tilde{x}_\alpha$ is the lift of $x_\alpha$ to $\tilde{\mathcal{M}}_0$. By Proposition 7.2.1, the curve $x_\alpha(t)$ is calibrated by the $n$-periodic viscosity solutions $v$ and $w$. Hence, using the non-negativity 7 of Proposition 5.2.14 and the liminf Properties (7.3.5), we get

$$\begin{aligned}0 \le h^{n\infty}(x_\alpha, x_\alpha) &\le \lim_i h^{n(k_{i+1}-k_i)}(x(-nk_{i+1}), x(-nk_i)) \\ &= \lim_i v(-nk_i, x(-nk_i)) - v(-nk_{i+1}, x(-nk_{i+1})) \\ &= \lim_i v(0, x(-nk_i)) - v(0, x(-nk_{i+1})) \\ &= v(x_\alpha) - v(x_\alpha) = 0\end{aligned}$$

Thus, $x_\alpha$ belongs to $\mathcal{M}_n$. Therefore, we obtain that $v(x_\alpha) = w(x_\alpha)$, and by calibration

$$\begin{aligned} w(x) &= w(-nk_i, x(-nk_i)) + h^{nk_i}(x(-nk_i), x) \\ &= w(x(-nk_i)) + h^{nk_i}(x(-nk_i), x) \\ &= \lim_i w(x(-nk_i)) + h^{nk_i}(x(-nk_i), x) \\ &= \lim_i w(x_\alpha) + h^{nk_i}(x(-nk_i), x) \\ &= \lim_i v(x_\alpha) + h^{nk_i}(x(-nk_i), x) \\ &= \lim_i v(x(-nk_i)) + h^{nk_i}(x(-nk_i), x) = v(x) \end{aligned}$$

which yields $v_{|\mathcal{M}_0} = w_{|\mathcal{M}_0}$ and by the Uniquenesse Theorem 7.2.4, $v = w = \Phi_n \circ \Psi_n(v)$. □

**Remark 7.4.5.** 1. We can derive from this proof and that of Theorem 7.2.4 the following uniqueness theorem for periodic viscosity solutions

**Proposition 7.4.6.** *For any viscosity solutions $v \in \mathrm{Per}_n(\mathcal{T})$ and $w \in \Omega(\mathcal{T})$, if $v_{|\mathcal{M}_n} = w_{|\mathcal{M}_n}$, then $v = w$.*

2. The generalization presented here can be considered a direct application of the representation of weak-KAM solutions demonstrated in [CISM13], as $n$-periodic solutions can be viewed as weak-KAM solutions of the Hamiltonian $H_n = nH(nt, x, p)$.

This theorem allows us to more easily describe the non-wandering set of systems in which every element of the Mather set is periodic with a uniform integer period, as stated in Corollary 7.1.5.

*Proof of Corollary 7.1.5.* 7.1.5. It suffices to prove that for any $v \in \Omega(\mathcal{T})$, $v_{|\mathcal{M}_0}$ is $N$-dominated on $\mathcal{M}_0$. Let $x$ and $y$ be two points of $\mathcal{M}_0$. Let $\tilde{y}$ be the lift of $y$ in $\tilde{\mathcal{M}}_0$ and set $y(t) = \pi \circ \phi_L^t(\tilde{y})$. If $u$ is a weak-KAM solution, we know from Proposition 7.2.1 that $y(t)$ is calibrated by both $u$ and $v$. Hence, for all integer $k \geq 0$, we have

$$\begin{aligned} v(kN, y) - v(0, y) &= v(kN, y(kN)) - v(0, y) = h^{kN}(y, y(kN)) \\ &= u(kN, y(kN)) - u(0, y) = u(0, y) - u(0, y) = 0 \end{aligned}$$

where we used the $N$-periodicity of $y(t)$ and the 1-time periodicity of the weak-KAM solution $u$. Thus, by definition of viscosity solutions, we get

$$v(y) - v(x) = v(kN, y) - v(x) \leq h^{kN}(x, y)$$

and taking the liminf on $k$, we conclude that

$$v(y) - v(x) \leq h^{N\infty}(x, y)$$

and in particular, $v_{|\mathbb{M}_N}$ belongs to $\mathrm{Dom}(\mathbb{M}_N, h^{N\infty})$. Therefore, $v = \Psi_N(v_{|\mathbb{M}_N})$ belongs to $\mathrm{Per}_N(\mathcal{T})$. and we conclude that $\Omega(\mathcal{T}) = \mathrm{Per}_N(\mathcal{T})$. □

### 7.4.3 $\underline{p}$-Recurrent Viscosity Solutions

In this section, we fix a sequence $\underline{p}$ of increasing positive integers and we discuss to which extend it is possible to represent $\underline{p}$-recurrent viscosity solutions.

#### $\underline{\mathrm{p}}$-Domination

We define the natural domination notion associated to $\underline{p}$-recurrence.

**Definition 7.4.7.** a map $\psi : X \to \mathbb{R}$ is said $\underline{p}$-*dominated* in a set $X \subset M$ if it is $h^{\underline{p}}$-dominated.

Then we have the following

**Proposition 7.4.8.** *Let $v \in \Omega(\mathcal{T})$ be a $\underline{p}$-recurrent viscosity solution. Then $v$ is $\underline{p}$-dominated in $M$.*

*Proof.* By definition of viscosity solution, we have for any $x$ and $y$ in $M$

$$v(y) - v(x) = \lim_n v(p_n, y) - v(x) \leq \liminf_n h^{p_n}(y, x) = h^{\underline{p}}(x, y)$$

□

**Remark 7.4.9.** (Failure of Reprenting $\underline{p}$-recurrent viscosity solutions)

1. (Failure of Uniqueness) Unlike what was observed for periodic viscosity solutions in Remark 7.4.5, there is no uniqueness theorem in the set

$$\mathcal{M}_{\underline{p}} := \{x \in \mathcal{M}_0 \mid h^{\underline{p}}(x, x) = 0\} \tag{7.4.11}$$

Hence, if one hopes to get a representation formula of $\underline{p}$-recurrent viscosity solutions, it must be on the entire Mather set $\mathcal{M}_0$.

2. (Failure of $\underline{p}$-recurrence for $h^{\underline{p}}$) If we define the injective map

$$\begin{array}{cccc} \tilde{\Psi}_{\underline{p}} \, \mathrm{Dom}(\mathcal{M}_0, h^{\underline{p}}) & \longrightarrow & \Omega(\mathcal{T}) \\ \psi & \longmapsto & \inf_{y \in \mathcal{M}_0} \{\psi(y) + h^{\underline{p}}(y, \cdot)\} \end{array} \tag{7.4.12}$$

Then, by the $\underline{p}$-domination Proposition 7.4.8, we get that all $\underline{p}$-recurrent maps belong to the image $\tilde{\Psi}_{\underline{p}}(\mathrm{Dom}(\mathcal{M}_0, h^{\underline{p}}))$. However, the $\underline{p}$-recurrence of $h^{\underline{p}}(x, \cdot)$ does not generally hold if $x$ is not $\underline{p}$-recurrent under the projected Lagrangian flow. Even if this condition were satisfied, there is no guarantee that the infimum $v$ of $\underline{p}$-recurrent viscosity solutions $v_n$ would be $\underline{p}$-recurrent, unless there is uniform $\underline{p}$-recurrence for $v_n$.

Nevertheless, a representation formula using the $\underline{p}$-barrier $h^{\underline{p}}$ is possible when it coincides with the general barrier $\underline{h}$ defined in Subsection 7.3.1. In the next subsections, we will consider cases where we can choose $\underline{h} = \underline{h}^p$.

### The Representation Formula for Mather Sets with $\underline{p}$-Recurrent Elements

In this section, we assume that there exists a sequence $\underline{p}$ of increasing positive integers for which the Mather set $\tilde{\mathcal{M}}_0$ verifies

$$\forall \tilde{x} \in \tilde{\mathcal{M}}_0, \quad \lim_n \phi_L^{-p_n}(\tilde{x}) = \tilde{x} \tag{7.4.13}$$

**Proposition 7.4.10.** *In this case, for all $x$ and $y$ in $\mathcal{M}_0$ and $z$ in $M$*

$$h^{\underline{p}}(x,z) \le h^{\underline{p}}(x,y) + h^{\underline{p}}(y,z) \tag{7.4.14}$$

*and the following equality holds*

$$\underline{k} = h^{\underline{p}}_{|\mathcal{M}_0 \times M} \tag{7.4.15}$$

*Proof.* We choose $\underline{h}(x,y) = h^{\underline{p}}(x,y)$. Fix three $x$, $y$ in $\mathcal{M}_0$ and $z$ in $M$. The triangular inequality (5.2.10) on the potential $h$ gives for all integer $n \ge 0$,

$$h^{\underline{p}+p_n}(x,z) \le h^{\underline{p}}(x,y) + h^{p_n}(y,z)$$

We know from Property 4 of Proposition 7.3.5 that the viscosity solution $\underline{h}(x,\cdot) = h^{\underline{p}}(x,\cdot)$ is $\underline{p}$-recurrent. Thus, taking the liminf on $n$ in the preceding inequality yields

$$h^{\underline{p}}(x,z) = \lim_n h^{\underline{p}+p_n}(x,z) \le h^{\underline{p}}(x,y) + \liminf_n h^{p_n}(y,z) = h^{\underline{p}}(x,y) + h^{\underline{p}}(y,z)$$

The triangular inequality results in $\underline{h} \le \underline{k}$ on $\mathcal{M}_0^R \times M = \mathcal{M}_0 \times M$. And since the inverse inequality is given by (7.3.21), we deduce the equality $h^{\underline{p}} = \underline{h} = \underline{k}$ on $\mathcal{M}_0 \times M$. □

Following this proposition, we can apply Theorem 7.1.2 to obtain

**Theorem 7.4.11.** *In this case, the representation formula is given by the following bijection*

$$\begin{array}{rccc} \Psi_{\underline{p}} : \mathrm{Dom}(\underline{\mathbb{M}}, h^{\underline{p}}) & \longrightarrow & \Omega(\mathcal{T}) \\ \psi & \longmapsto & \inf_{y \in \underline{\mathbb{M}}} \{\psi(y) + h^{\underline{p}}(y,\cdot)\} \end{array} \tag{7.4.16}$$

*with its inverse being the restriction map*

$$\begin{array}{rccc} \Phi_{\underline{p}} : \Omega(\mathcal{T}) & \longrightarrow & \mathrm{Dom}(\underline{\mathbb{M}}, h^{\underline{p}}) \\ v & \longmapsto & v_{|\underline{\mathbb{M}}} \end{array} \tag{7.4.17}$$

**Remark 7.4.12.** 1. Note that, even if every recurrent viscosity solutions $v \in \Omega(\mathcal{T})$ are expressed as the infimum of $\underline{p}$-recurrent ones, it does not imply that $v$ is itself $\underline{p}$-recurrent. This is due to the non-uniform $\underline{p}$-recurrent of the viscosity solutions $h^{\underline{p}}(x,\cdot)$ for $x \in \mathcal{M}_0$.

Finding a sequence $\underline{q}$ such that $v$ is $\underline{q}$-recurrent remains highly non-trivial, even in this case.

2. It is possible to verify through the proof of Theorem 7.1.2 that this formula remains valid if we assume that for all $\tilde{x} \in \tilde{\mathcal{M}}_0$, either $\lim_n \phi_L^{+p_n}(\tilde{x}) = \tilde{x}$ or $\lim_n \phi_L^{-p_n}(\tilde{x}) = \tilde{x}$. Positive or negative time recurrence does not matter for $h^{\underline{p}}$ to satisfy the triangular inequality. Hence, $d_{\underline{p}}(x,y) = h^{\underline{p}}(x,y) + h^{\underline{p}}(y,x)$ is a pseudometric on $\mathcal{M}_0$, making it possible to define its associated equivalence relation $\sim_{\underline{p}}$ and $\underline{p}$-static classes $\mathbb{M}_{\underline{p}}$. Since a uniqueness result on $\mathbb{M}_{\underline{p}}$ holds, the remainder of the proof follows analogously.
3. This Representation Formula is still valid if we only had $\phi_L^1$-recurrence on a dense subset $\mathcal{M}_0'$ of $\mathcal{M}_0$. In this case, we would still have by continuity that $\underline{k} = h^{\underline{p}}$ and a uniqueness theorem on $\mathcal{M}_0'$.
4. Following this last remark, this case includes the situation where the Mather set contains a dense set of periodic curves with integer periods. If $q_n$ is the sequence of these periods, we take $p_n = \prod_{k=0}^n q_k$.

### The Representation Formula for Mather Sets with Uniformly $\underline{p}$-Recurrent Elements

We assume that there exists an increasing sequence of positive integers $\underline{p}$ such that $\phi^{-p_n}_{L|\tilde{\mathcal{M}}_0}$ uniformly converges to the identity.

**Proposition 7.4.13.** *In this case, the maps $h^{\underline{p}}(x,\cdot)$ are uniformly $\underline{p}$-recurrent. Their convergence speed is controlled by the convergence of $\phi^{-p_n}_{L|\tilde{\mathcal{M}}_0}$ to the identity. More precisely, if for any $y \in \mathcal{M}_0$ with lift $\tilde{y}$ in $\tilde{\mathcal{M}}_0$ we set the curve $y(t) = \pi \circ \phi_L^t(\tilde{y})$, then*

$$\|\underline{h}^{p_n}(x,\cdot) - \underline{h}(x,\cdot)\|_\infty \leq 2\kappa_1 . \sup_{y \in \mathcal{M}_0} d(y(-p_n), y) \tag{7.4.18}$$

*Proof.* This is a direct consequence of identity (7.3.8). □

The representation formula results in the following corollary which implies Corollary 7.1.6.

**Corollary 7.4.14.** *All the elements $v$ of $\Omega(\mathcal{T})$ are $\underline{p}$-recurrent and*

$$\|v(p_n,\cdot) - v\|_\infty \leq 2\kappa_1 . \sup_{y \in \mathcal{M}_0} d(y(-p_n), y) \tag{7.4.19}$$

*Proof.* Let $v$ be an element of $\Omega(\mathcal{T})$. By the Representation Theorem 7.4.11, there exists a $\underline{p}$-dominated map $\psi \in \mathrm{Dom}(\underline{\mathbb{M}}, h^{\underline{p}})$ such that $v = \Psi_{\underline{p}}(\psi)$. Then, Proposition 6.2.1 shows that for all positive integer $n$

$$\begin{aligned}\mathcal{T}^n v(x) &= \inf_{y \in \underline{\mathbb{M}}} \left\{\psi(y) + \mathcal{T}^n h^{\underline{p}}(y, \cdot)(x)\right\} \\ &= \inf_{y \in \underline{\mathbb{M}}} \left\{\psi(y) + h^{\underline{p}+n}(y, x)\right\}\end{aligned}$$

Moreover, we infer from Proposition 7.4.13 that for all $x \in M$, $y \in \mathcal{M}_0$ and $n \geq 0$

$$\left|[\psi(y) + h^{\underline{p}+p_n}(y,x)] - [\psi(y) + h^{\underline{p}}(y,x)]\right| = |h^{\underline{p}+p_n}(y,x) - h^{\underline{p}}(y,x)| \leq 2\kappa_1. \sup_{z \in \mathcal{M}_0} d(z(p_n), z)$$

where the bound is uniform on $y \in \mathcal{M}_0$. Taking the infimum on $y$ yields

$$\|v(p_n) - v\|_\infty \leq 2\kappa_1. \sup_{z \in \mathcal{M}_0} d(z(p_n), z) \longrightarrow 0 \quad \text{as } n \to \infty$$

and $v$ is a $\underline{p}$-recurrent viscosity solution. □

# Chapitre 8

# A Recurrent, Non-Periodic and Smooth Viscosity Solution of the Hamilton-Jacobi Equation

In the autonomous case, the asymptotic behavior of the Lax-Oleinik operator $\mathcal{T}$ has been extensively studied in [Roq98, NR97a, NR97b, NR99]. This behavior is well understood, primarily due to Albert Fathi's convergence theorem [Fat98], which demonstrates the convergence of viscosity solutions to steady states known as weak-KAM solutions. These solutions are the fixed points of the Lax-Oleinik operator $\mathcal{T}$. We refer to [BS00] for an analytical proof.

However, addressing the time-dependent framework reveals more challenging. According to A. Fathi and J.N. Mather [FM00], such convergence does not hold in the non-autonomous setting. Furthermore, in the one-dimensional case ($M = \mathbb{T}^1$), P. Bernard and J.-M. Roquejoffre [Roq98, BR04] demonstrated that viscosity solutions converge, up to a linear time dependence, to periodic solutions, thereby constraining the possible behaviors in dimension one. Their work elucidates that the periodicity of asymptotic viscosity solutions is intricately linked to the periodicity of the orbits within the Mather set. This set is a dynamical subset of the phase space emerging from Aubry-Mather theory, and plays a central role in weak-KAM theory.

Less is known about higher dimensions. This work presents the construction of a smooth, recurrent, non-periodic viscosity solution of the Hamilton-Jacobi equation (5.1.8) on any manifold $M$ of dimension 2 or higher, thereby refuting any possible generalization of the result by Bernard and Roquejoffre to higher dimensions.

The main idea is to make advantage of the correlation between periodic orbits within the Mather set and periodic viscosity solutions. Specifically, we defined a an increasingly diverging sequence of periodicities $\rho_n$ and, for each of them, a $\rho_n$-periodic orbit $\{x_n^i\}_i$ within the Mather set, and a $\rho_n$-periodic viscosity solution through modified Peierls Barriers $h^{\rho_n\infty}(x_n,\cdot)$ (introduced in [FM00]). The chosen initial data $u$ for a recurrent, non-periodic viscosity solution was of the form $u = \inf_{n\geq 0}\{h^{\rho_n}(x_n,\cdot)\}$.

This idea allows the construction of numerous examples. However, achieving more than Lipschitz regularity for the recurrent solution reveals more intricate. To address this, adjustments are made to the Hamiltonian and the viscosity solution $u$ to achieve $C^\infty$ regularity with a new initial data of the form $u = \inf_{n\geq 0}\{c_n + h^{\rho_n}(x_n,\cdot)\}$ where $c_n$ are constants to be chosen carefully. This yields the following theorem.

**Theorem 8.0.1.** *For any closed manifold $M$ of dimension $d \geq 2$, there exists a $C^\infty$ Tonelli Hamiltonian $H : \mathbb{T}^1 \times T^*M \to \mathbb{R}$ such that the Lax-Oleinik operator $\mathcal{T}$ admits a $C^\infty$ recurrent, non-periodic viscosity solution.*

After the initial construction, our focus shifts to analyzing the asymptotic behavior of the Lax-Oleinik operator for the obtained Tonelli Hamiltonian. We identify the $\omega$-limit set $\omega(u)$ of the constructed viscosity solution $u$ and, more broadly, describe the entire non-wandering set $\Omega(\mathcal{T})$ using a generalized representation formula from Theorem 7.1.2. In this context, we observe that when the Mather set consists solely of periodic orbits with integer periods, it imparts odometer-like dynamics to the non-wandering set (see [HR79, Dow05, HV22] for surveys on odometers, also known as adding machines). This study leads to the following theorem.

**Theorem 8.0.2.** *Assume that the Mather set $\mathcal{M}$ is the union of periodic orbits with integer periods. Then for all $v \in \Omega(\mathcal{T})$, the restriction of the Lax-Oleinik operator $\mathcal{T}$ to the set $\omega(v)$ is a factor of an odometer.*

It is known that the Lax-Oleinik semigroup $\mathcal{T}$ acts by isometries on its non-wandering set. This imposes limitations on the possible dynamics that can occur on $\omega(u)$ for any recurrent viscosity solution $u$. A natural question that arises is of...

**Question 8.0.3.** Constructing an explicit Tonelli Hamiltonian $H$ for which the action of the Lax-Oleinik operator $\mathcal{T}$ on its non-wandering set $\Omega(\mathcal{T})$ exhibits more intricate dynamics than that of an odometer.

In Section 8.1, following ideas from Mañé, we construct a Tonelli Hamiltonian for which $\Omega(\mathcal{T})$ contains a smooth, non-wandering, non-periodic viscosity solution. Section 8.2 is dedicated to the main construction of a recurrent, non-periodic, yet non-regular viscosity solution $u$, along with a description of its $\omega$-limit set. We then extend this description to the non-wandering set of the Lax-Oleinik semigroup $\mathcal{T}$. Finally, in Section 8.3, we adjust the initial condition to obtain a smooth, recurrent, non-periodic viscosity solution.

## 8.1 Construction of the Mañé Hamiltonian

In this section, we construct a Tonelli Hamiltonian $H : \mathbb{T}^1 \times T * M \to \mathbb{R}$ whose Hamilton-Jacobi equation admits a smooth, recurrent, non-periodic viscosity solution. This construction is inspired by Mañé's approach in the appendix of [Mn92].

The idea of the construction involves creating regions in $M$ where a selected viscosity solution is periodic with a prescribed minimal period. Taking these periods to infinity will make the solution non-periodic, and controlling the size of the regions facilitates the proof of the recurrence.

We will also highlight certain symmetries imposed on the Hamiltonian vector field, which are crucial for ensuring the existence of regular, recurrent, non-periodic viscosity solutions, as will be shown in Section 8.3.

In the first subsection, we present the construction of the Hamiltonian, and in the second, we analyze the behavior of the $n$-Peierls barriers $h^{n\infty}$ (see Definition 7.4.3), which will be essential for building periodic viscosity solutions with prescribed minimal periods.

### 8.1.1 Construction of the Lagrangian $L$

We construct the Lagrangian $L$ following the examples constructed by Mañé in the appendix of [Mn92]. We will designate these Lagrangians as Mañé Lagrangians.

The idea is to select the desired dynamics of $f_t = \phi_L^t$ on a specific submanifold of $TM$ and subsequently extending it into a Lagrangian flow across the entire tangent bundle. From the Hamiltonian perspective, the initial submanifold corresponds to the zero-section of $T^*M$ which yields the desired Hamiltonian dynamics.

#### Subdivision of the Manifold $M$

We first need to fix a region of work in the manifold $M$ where we will define the isotopy $f_t$ as desired. We work on a fixed chart $U \to \mathbb{R}^d$ of $M$. This chart will be identified to an open 0-centered ball $B$ of $\mathbb{R}^d$. We use the coordinates $(r, \theta, x_3, ..., x_d)$ in $\mathbb{R}^d$ where

$(r,\theta) \in \mathbb{R}_{\geq 0} \times \mathbb{T}^1$ are the polar coordinates on the plane $\mathbb{R}^2 \times \{0\}$ and $(x_3, ..., x_d)$ are the remaining canonical coordinates of $\{0\} \times \mathbb{R}^{d-2}$. The space $\mathbb{R}^d$ is endowed with its canonical $L^2$-metric $\|v\| = \sqrt{\sum_{k=1}^d v_k^2}$ and its relative distance $d$. The manifold $M$ is endowed with a Riemannian metric so that the chart $U \to \mathbb{R}^d$ is a Riemannian isometry onto its image.

We aim for the map $f = f_1$ to possess a family of attractive orbits $\{x_n^i\}_{0\leq i\leq \rho_n - 1}$ with increasing periodicities $\rho_n$, each having distinct attraction basins in the sense that no $\varepsilon$-orbit can transition from one basin to another.

Let $(\rho_n)_{n\geq 0}$ be an increasing sequence of positive integers with $\rho_0 \geq 2$. Let $(r_n)_{n\geq 0}$ be a decreasing sequence of positive radii, converging to 0. We take $r_0$ smaller than the radius of the ball $B$ defining the chart of $M$. Consider the circles $O_n = \{(r_n, \theta, 0, .., 0) \mid \theta \in \mathbb{T}^1\} \subset \mathbb{R}^2 \times \{0\}$ and set the $n$-th orbit to be $x_n^i = (r_n, \frac{i}{\rho_n}, 0, .., 0) \in O_n$ for $i = 0, .., \rho_n - 1$.

Let $(\delta_n)_{n\geq 0}$ be a decreasing sequence of sufficiently small positive real numbers. And consider the following different sets

$$\begin{aligned}
O_n &= \{(r_n, \theta, 0, .., 0) \mid \theta \in \mathbb{T}^1\} \\
B_n^i &= \{x \in \mathbb{R}^d \mid d(x, x_n^i) < \delta_n\} \\
B_n &= \{x \in \mathbb{R}^d \mid d(x, O_n) < \delta_n\} \\
A_n &= \{x \in \mathbb{R}^d \mid \delta_n < d(x, O_n) < 2\delta_n\} \\
C_n &= A_n \cup \overline{B_n} = \{x \in \mathbb{R}^d \mid d(x, O_n) < 2\delta_n\} \\
D_n &= \{x \in \mathbb{R}^d \mid 2\delta_n < d(x, O_n) < 3\delta_n\} \\
D &= M \setminus \Big(\bigcup_{n=0}^{\infty} \overline{C_n}\Big) \\
D' &= M \setminus \Big(\bigcup_{n=0}^{\infty} \overline{C_n \cup D_n}\Big)
\end{aligned} \tag{8.1.1}$$

The radii $\delta_n$ are taken small enough so that all of these defined sets are in the chart $B$, all the closed balls $(\overline{B_n^i})_i$ are disjoint and all the closed sets $(\overline{C_n \cup D_n})_{n\geq 0}$ are disjoint.

**Remark 8.1.1.** Note that in dimension 2, the sets $A_n$ have two connected components contrasting with the higher dimensional case. We denote these connected components by $A_n^{\pm}$ where $A_n^+$ corresponds to the larger $r$-coordinate, and $A_n^-$ to the lower. We also define $D_n^{\pm}$ in the same way.

This distinction will take more importance in the study of the regularity in Section 8.3 where we will need to separate the two cases.

We consider the following points

$$\begin{aligned}
x_n^i &= (r_n, \frac{i}{\rho_n}, 0, .., 0) \in O_n \\
y_n &= (r_n + \delta_n, 0, 0, .., 0) \in \partial B_n \\
z_\infty &= (0, 0, .., 0) \in \overline{D} \\
z_n^\pm &= (r_n \pm 2\delta_n, 0, 0, .., 0) \in \partial A_n^\pm \cap \partial C_n \text{ in the 2D case}
\end{aligned} \tag{8.1.2}$$

The different sets of (8.1.1) and points of (8.1.2) are represented in the Figure 8.1 down below.

### The Isotopy $f_t$

Now we define the isotopy $f_t$ as the composition of a "radial" autonomous Lagrangian flow $g_t$ with a rotational isotopy $\mathcal{R}_t$.

**The Isotopy $g_t$.** Let $(\varsigma_n)_n$ be a sequence of small real numbers, let $\chi : \mathbb{R} \to \mathbb{R}_{\geq 0}$ be a smooth bump function with support in $[0,1]$ and positive on $(0,1)$. And let $p_{C_n} : A_n \to \partial C_n$ be the projection on $\partial C_n$ defined as follows

$$p_{C_n}(x) = p_{O_n}(x) + 2\delta_n . \frac{x - p_{O_n}(x)}{||x - p_{O_n}(x)||} \quad \text{where } p_{O_n}(x) = (\rho_n, \theta_x, 0, .., 0) \tag{8.1.3}$$

We define the autonomous vector field $Z : M \to TM$ by

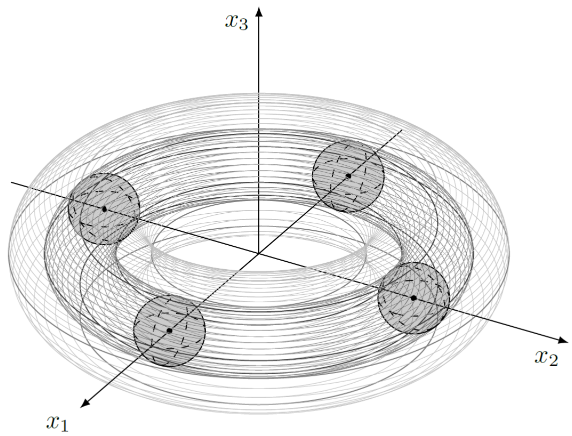
$$Z(x) = \begin{cases} -\varsigma_n . \chi(||x - x_n^i||^2/\delta_n^2).(x - x_n^i) & \text{if } x \in B_n^i \\ -\varsigma_n . \chi(||x - p_{C_n}(x)||^2/\delta_n^2).(x - p_{C_n}(x)) & \text{if } x \in A_n \\ 0 & \text{elsewhere} \end{cases} \tag{8.1.4}$$

and let $(g_t)_{t \in [0,1]}$ be the associated flow.

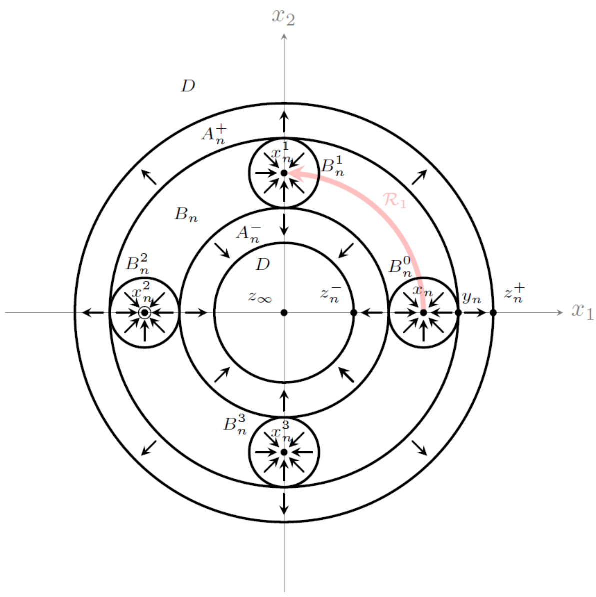
The restriction of $g := g_1$ to $B_n^i$ is radially symmetric with respect to the center $x_n^i$ which is attractive. Similarly, the restriction of $g$ to $A_n$ has a radial symmetry with respect to the circle $O_n$ meaning that for a fixed angle $\theta$, the restriction of $g$ to the "hollowed" ball $A_n \cap \{\theta_x = \theta\}$ is symmetric with respect to its center $O_n \cap \{\theta_x = \theta\}$. Moreover, the set $\partial C_n$ is attractive for its dynamics. This can be observed in Figure 8.1.

Computation shows that the vector field $Z$ is $C^k$-regular if $\varsigma_n \underset{n \to \infty}{=} o(\delta_n^k)$.

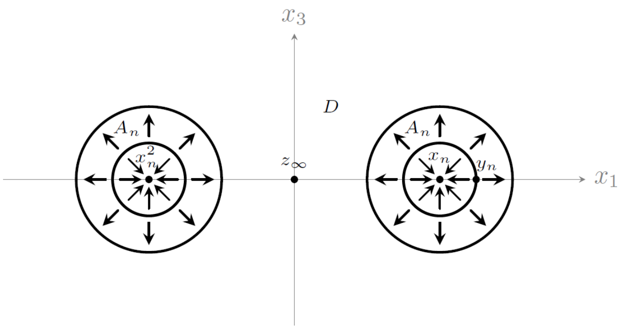
**The Isotopy $\mathcal{R}_t$.** Let $\eta : \mathbb{R} \to \mathbb{R}$ be a smooth map, null on $(-\infty, 0]$, constant equal to 1 on $[1, +\infty)$ and increasing on $[0,1]$. And consider for all integer $n \geq 0$ the map $\eta_n : D_n \to [0,1]$




(a) 3D View of $C_n$.




(b) Top View of $C_n$ / The 2D Case.




(c) Side View of $C_n$.

FIGURE 8.1 – The dynamics of the flow $g_t$ on the set $C_n$

defined by

$$\eta_n(x) = 1 - \eta(\|x - p_{C_n}(x)\|^2/\delta_n^2) \tag{8.1.5}$$

For all angles $\alpha \in \mathbb{T}^1$, we define the rotation $R_\alpha : \mathbb{R}^d \to \mathbb{R}^d$ by

$$R_\alpha(r, \theta, x_3, .., x_d) = (r, \theta + \alpha, x_3, .., x_d) \tag{8.1.6}$$

The isotopy $(\mathcal{R}_t)_{t \in [0,1]}$ is taken as

$$\mathcal{R}_t(x) = \begin{cases} R_{\frac{1}{\rho_n}\eta(t)}(x) & \text{if } x \in \overline{C_n} \\ \left(r, \theta + \frac{\eta(t)}{\rho_n}\eta_n(x), x_3, .., x_d\right) & \text{if } x \in D_n \\ x & \text{elsewhere} \end{cases} \tag{8.1.7}$$

It is a rotation of constant angle $\rho_n$ on the sets $C_n$, and these angles decrease radially and progressively on $D_n$ until they reach zero on $D'$.

Computation shows that the map $\mathcal{R}_t$ is $C^k$-regular if $\frac{1}{\rho_n} \underset{n\to\infty}{=} o(\delta_n^k)$.

**The Isotopy $f_t$.** The desired isotopy $(f_t)_{t \in [0,1]}$ is the composition

$$f_t = \mathcal{R}_t \circ g_t \tag{8.1.8}$$

**Remark 8.1.2. Symmetries of $\mathcal{R}_t$ and $g_t$.** Notice that

- In a neighbourhood of the sets $\overline{C_n}$, the isotopies $\mathcal{R}_t$, $g_t$ and hence $f_t$ commutes with the rotation $R_{\frac{i}{\rho_n}}$ for every $0 \leq i < \rho_n$.
- In the balls $B_n^i$, the isotopy $g_t$ is symmetric with respect to the center $x_n^i$ of $B_n^i$ meaning that it commutes with all affine isometries of the form $x_n^i + U$ where $U \in O(n)$.
- In the sets $A_n$, the isotopies $\mathcal{R}_t$, $g_t$ and hence $f_t$ are symmetric with respect to the circles $O_n$ meaning that for any fixed angle $\theta$, they commute with all the affine isometries defined on the set $A_n \cap \{\theta_x = \theta\}$ and of the form $x_n^\theta + U$ where $x_n^\theta = (\rho_n, \theta, 0, .., 0) \in O_n$ and $U \in O(n-1)$. Additionally, in the sets $A_n$, they commute with the translations in the $\theta$-coordinate.

**Remark 8.1.3.** 1. Computation shows that the smaller the quantities $\frac{1}{\rho_n}$ and $\varsigma_n$ are, the closer is the map $f = f_1$ to the identity $Id_M$ in the $C^\infty$-topology.

2. As mentioned for the maps $g_t$ and $\mathcal{R}_t$, the map $f_t$ is $C^{k+1}$-regular provided that

$$\varsigma_n \underset{n\to\infty}{=} o(\delta_n^k) \quad \text{and} \quad \frac{1}{\rho_n} \underset{n\to\infty}{=} o(\delta_n^k) \tag{8.1.9}$$

Assuming that these identities hold for all integers $k \geq 0$, we obtain a smooth isotopy $f_t$.

### The Mañé Lagrangian $L$

The Mañé Lagrangian $L$ is defined using the vector field $X_t$ of the constructed isotopy $f_t$. Additionally, to ensure periodicity in time for $L$, it is essential to confirm that the vector field $X_t$ is itself time-periodic.

**The Vector Field $X_t$.** The first point of Remark 8.1.3 tells us that if $\frac{1}{\rho_n}$ and $\delta_n$ are small enough, we get for all times $t \in [0,1]$

$$||f_t - Id_M||_{C^1} < 1 \tag{8.1.10}$$

and in particular, the map $f_t$ is invertible for all times $t$.

This allows us to define the corresponding vector field $X(t,x) = X_t(x) : \mathbb{R} \times M \to TM$ by

$$X_t = \frac{df_t}{dt} \circ f_t^{-1} \tag{8.1.11}$$

We know that for all positive integer $k \geq 1$, $\eta^{(k)}(0) = \eta^{(k)}(1) = 0$. Thus, $X_0 = X_1 = 0$ and so are all of its time derivatives. This implies that the vector field $X_t$ reduces to be a smooth time periodic vector field $X_t : \mathbb{T}^1 \times M \to TM$.

Consider the time periodic vector fields $Y_t$ relative to the isotopy $\mathcal{R}_t$ and recall the relation between $Z$ and $g_t$

$$Y_t = \frac{d\mathcal{R}_t}{dt} \circ \mathcal{R}_t^{-1} \quad \text{and} \quad Z = \frac{dg_t}{dt} \circ g_t^{-1} \tag{8.1.12}$$

**Proposition 8.1.4.** *We have*

$$X_t = Y_t + d\mathcal{R}_t.Z \circ \mathcal{R}_t^{-1} \tag{8.1.13}$$

*Proof.* We have

$$\frac{df_t}{dt} = \frac{d(\mathcal{R}_t \circ g_t)}{dt} = \frac{d\mathcal{R}_t}{dt} \circ g_t + d\mathcal{R}_t.\frac{dg_t}{dt}$$

Then

$$\begin{aligned} X_t = \frac{df_t}{dt} \circ f_t^{-1} &= \frac{d\mathcal{R}_t}{dt} \circ g_t \circ g_t^{-1} \circ \mathcal{R}_t^{-1} + d\mathcal{R}_t.\frac{dg_t}{dt} \circ g_t^{-1} \circ \mathcal{R}_t^{-1} \\ &= Y_t + d\mathcal{R}_t.Z \circ \mathcal{R}_t^{-1} \end{aligned}$$

□

**Remark 8.1.5. Symmetries of $X_t$.** We can deduce from the symmetries of $\mathcal{R}_t$ and $g_t$ stated in Remark 8.1.2 that

- In a neighbourhood of the sets $\overline{C_n}$, the vector field $X_t$ is $R_{\frac{i}{\rho_n}}$-invariant for every $0 \le i < \rho_n$.
- In the balls $B_n^i$, the vector field $Z$ is symmetric with respect to the center $x_n^i$ of $B_n^i$.
- In the sets $A_n$, the vector field $X_t$ is symmetric with respect to the circles $O_n$ and with respect to the $\theta$-coordinate.
- In the sets $D_n$, we have equality $X_t = Y_t$.
- In the set $D'$, the vector field $X_t$ is null.

These symmetries will show crucial in the study of regularity in Section 8.3.

**The Mañé Lagrangian $L$.** Following the appendix of [Mn92], we define a smooth time-periodic Mañé Lagrangian $L : \mathbb{T}^1 \times TM \to \mathbb{R}$ by

$$L(t,x,v) = \frac{1}{2}\|v - X_t(x)\|^2 \tag{8.1.14}$$

where $\|.\|$ is the norm coming from the metric of $M$. It reduces to the $L^2$-norm when restricted to the chart where the dynamics are non trivial.

**Proposition 8.1.6.** *The Mañé Hamiltonian $H : \mathbb{T}^1 \times T^*M \to \mathbb{R}$ associated to the Mañé Lagrangian $L$ is given by*

$$H(t,x,p) = \frac{1}{2}\|p + X_t(x)\|^2 - \frac{1}{2}\|X_t(x)\|^2 \tag{8.1.15}$$

*with vector field $X_H = (X_H^x, X_H^p)$ given by*

$$X_H^x(t,x,p) = p + X_t(x) \quad \text{and} \quad X_H^p(t,x,p) = -p.dX_t(x) \tag{8.1.16}$$

*Proof.* Using the metric $\langle\cdot,\cdot\rangle$ induced by the norm $\|\cdot\|$, we identify elements of $v$ of$T_xM$ with elements $p$ of $T_x^*M$ as follows : $p = \langle v, \cdot\rangle$. Then, we have for all $(t,x) \in \mathbb{R} \times M$ and $(v,p) \in T_xM \times T_x^*M$

$$\begin{aligned} 2.\big[H(t,x,p) + L(t,x,v) - p.v\big] &= \|p + X_t(x)\|^2 - \|X_t(x)\|^2 + \|v - X_t(x)\|^2 - 2p.v \\ &= \|p\|^2 + 2p.X_t(x) + \|v\|^2 - 2v.X_t(x) + \|X_t(x)\|^2 - 2p.v \\ &= \|p - v + X_t(x)\|^2 \ge 0 \end{aligned}$$

with equality if and only if $p = v - X_t(x) = \partial_v L(t,x,v)$.

Direct computation yields

$$X_H^x(t,x,p) = \partial_p H(t,x,p) = p + X_t(x) \quad \text{and} \quad X_H^p(t,x,p) = -\partial_x H(t,x,p) = -p.dX_t(x)$$

□

**Proposition 8.1.7.** *The Mañé Lagrangian $L$ defined in (8.1.14) is Tonelli.*

*Proof. Regularity.* is verified by the assumptions of Remark 8.1.3.

*Strict Convexity.* We denote by $\langle\cdot,\cdot\rangle$ the scalar product associated to the norm $\|.\|$ on $TM$. Then, for all $(t,x,v) \in \mathbb{T}^1 \times TM$, we have

$$\partial_v L(t,x,v) = \langle v - X_t(x), \cdot\rangle \quad \text{and} \quad \partial^2_{vv} L(t,x,v) = \langle\cdot,\cdot\rangle$$

It follows that $\partial^2_{vv} L(t,x,v) > 0$ for all $(t,x,v) \in \mathbb{T}^1 \times TM$.

*Superlinearity.* Since the norm is squared in the definition (8.1.14) of $L$, we get $\frac{L(t,x,v)}{\|v\|} \to \infty$ as $\|v\| \to \infty$.

*Completeness.* We will prove the completeness for the Hamiltonian flow $\phi_H$ which is conjugated to the Lagrangian flow $\phi_L$ by the Legendre map $\mathcal{L}$ introduced in Subsection 5.1.1. Note from the formula of the Hamiltonian vector field $X_H$ expressed in (8.1.16), its time-periodicity and from the compactness of the manifold $M$ that there exist two positive real numbers $C_1$ and $C_2 > 0$ such that for all $(s,x,p) \in \mathbb{T}^1 \times T^*M$, we have

$$\|X_H(s,x,p)\| \leq C_1\|p\| + C_2$$

We work locally on charts in $\mathbb{R}^d$. Fix a time $T > 0$. For all time $t$ such that $|t-s| \leq T$ and $\phi_H^{s,t}(x,p)$ is well defined and belongs to the chart, we have

$$\left\|\frac{d}{dt}\phi_H^{s,t}(x,p)\right\| = \|X_H(t,\phi_H^{s,t}(x,p))\| \leq C_1\|\phi_H^{s,t}(x,p)\| + C_2$$

Hence, we get

$$\begin{aligned}
\|\phi_H^{s,t}(x,p)\| &\leq \|\phi_H^{s,t}(x,p) - (x,p)\| + \|(x,p)\| \\
&= \left\|\int_s^t X_H(\tau,\phi_H^{s,\tau}(x,p))\,d\tau\right\| + \|(x,p)\| \\
&\leq \int_s^t \|X_H(\tau,\phi_H^{s,\tau}(x,p))\|d\tau + \|(x,p)\| \\
&\leq C_1\int_s^t \|\phi_H^{s,\tau}(x,p)\|d\tau + C_2.(t-s) + \|(x,p)\|
\end{aligned}$$

$$\leq C_1 \int_s^t \|\phi_H^{s,\tau}(x,p)\| d\tau + C_2 T + \|(x,p)\|$$

Applying Grönwall lemma yields

$$\left\|\phi_H^{s,t}(x,p)\right\| \leq (C_2 T + \|(x,p)\|).e^{C_1(t-s)} \leq (C_2 T + \|(x,p)\|).e^{C_1 T}$$

This compactness result for a fixed $T > 0$ allow to extend $\phi_H^{s,t}(x,p)$ to all times $t$ such that $|t - s| \leq T$. Since this holds for all $T > 0$, we deduce the completeness of the Hamiltonian vector field $X_H$, and by conjugacy, the completeness of the Lagrangian vector field $X_L$. □

### 8.1.2 Evaluation of the Peierls Barriers of Mañé Lagrangians

This subsection examines the behaviour of the Peierls barriers $h^\infty$ and $h^{k\infty}$, that are specific to Mañé Lagrangians. These will provide a straightforward way to estimate the values of the barriers between two points in $M$.

#### The 0-set of the Peierls Barriers on Pseudo-Orbits of Mañé Lagrangians

Prior to selecting a viscosity solution that addresses the main Theorem 8.0.1 of this chapter, one needs to be able to identify the Mather set and the Peierls set associated to the constructed Mañé Lagrangian $L$. This constitutes the objective of the current subsection, where our focus will be on understanding the 0-set of the Peierls Barriers.

We recall some properties of the Lagrangian $L(t,x,v) = \frac{1}{2}\|v - X_t(x)\|^2$ mentioned in the previous subsection.

- The projection on $M$ of the Euler-Lagrange flow $\phi_L^t$ restricted to the subset $\mathcal{L}_L := \{v = X_t(x)\} \subset TM$ corresponds precisely to the flow $f_t$ of $X_t$.
- The subset $\mathcal{L}_L$ is a graph over the zero-section $0_{TM} \simeq M$ and its Hamiltonian counterpart $\mathcal{L}_H \subset T^*M$ is the zero-section $0_{T*M} \simeq M$ of the cotangent bundle. Furthermore, the Hamiltonian flow $\phi_H^t$ restricted to the zero-section is precisely the flow $f_t$.

The following proposition justifies the choice of $\alpha_0 = 0$ and of considering the Hamilton-Jacobi equation (5.1.8).

**Proposition 8.1.8.** *The Mañé critical value $\alpha_0$ of Mañé Lagrangians of the form $L(t,x,v) = \frac{1}{2}\|v - X_t(x)\|^2$ is null.*

*Proof.* Recall that the Mañé critical value is the real

$$\alpha_0 = -\inf_\mu \left\{ \int_{\mathbb{T}^1 \times TM} L \, d\mu \right\} \leq 0$$

where the infimum is taken over compact supported probability measures $\mu$ invariant by the Euler-Lagrangian flow corresponding to $L$.
Following the proof of the Krylov–Bogolyubov theorem (see [Mat91]), there exists a $f_t$-invariant measure $\mu$ supported on the submanifold $\{(t,x,v) \mid v = X_t(x)\} \subset TM$. Integrating $L$ with respect to $\mu$ gives

$$-\alpha_0 \leq \int_{\mathbb{T}^1 \times T^*M} L \, d\mu = \frac{1}{2} \int_{\mathrm{Supp}(\mu)} \|v - X_t(x)\|^2 \, d\mu = 0$$

Therefore $\alpha_0 = 0$. □

We get the identifications $h = h_0$ and $\mathcal{T} = \mathcal{T}_0$. And since the Mañé Lagrangian $L$ is non-negative, we get the following.

**Corollary 8.1.9.** *For all $x, y \in M$, all times $s < t$ and all integers $n, k \geq 1$, $h^{s,t}(x,y) \geq 0$ and $h^{n\infty}(x,y) \geq 0$.*

**Remark 8.1.10.** In a general framework, we only have $h^\infty(x,x) \geq 0$, however for two distinct points $x$ and $y$, the value of $h^\infty(x,y)$ can potentially be negative. The exceptional non-negativity of the barriers highly simplifies the Aubry-Mather theory of Mañé Lagrangians.

We will discern the conditions under which the Peierls Barrier vanishes and when it remains positive. But first, we introduce the following definition.

**Definition 8.1.11.** For any $\varepsilon > 0$, any fixed time $\tau > 0$, and any two points $x$ and $y$ of $M$, we call a $(\varepsilon, \tau)$-*pseudo-orbit* of the flow of $X_t$ between $x$ and $y$ a finite family of curves $(\gamma_k : [S_k, T_k] \to M)_{0 \leq k \leq m}$ such that

i. $S_0 = 0$ and $\gamma_0(0) = x$.

ii. For all $0 \leq k \leq m$, $\dot{\gamma}_k(t) = X_t(\gamma_k(t))$.

iii. The real times $S_k$ and $T_k$ verify $T_k - S_k \geq \tau$ and $S_{k+1} = T_k \; mod \; 1$.

iv. For all $0 \leq k \leq m-1$, $d\big(\gamma_k(T_k), \gamma_{k+1}(S_{k+1})\big) < \varepsilon$ and $d\big(\gamma_m(T_m), y\big) < \varepsilon$.

**Proposition 8.1.12.** *Let $x$ and $y$ be two points of $M$ such that there exists an increasing real sequence of positive times $(t_n)_{n \geq 0}$ with $\lim_n t_n = +\infty$ and $\lim_n h^{t_n}(x,y) = 0$. Then for every $\varepsilon > 0$ and $\tau > 0$, there exists an $(\varepsilon, \tau)$-pseudo-orbit of the flow of $X_t$ between $x$ and $y$.*

*By contraposition, if for some fixed $\tau > 0$ there exists a constant $\varepsilon > 0$ such that no $(\varepsilon, \tau)$-pseudo-orbit of the flow of $X_t$ links $x$ to $y$, then $\liminf\limits_{t \to +\infty} h^t(x,y) > 0$.*

**Lemma 8.1.13.** *Let $x$ and $y$ be two points of $M$ such that $\lim_n h^{t_n}(x,y) = 0$ with $t_n \to +\infty$ i.e. such that there exists a sequence of minimizing curves $(\sigma_n : [0, t_n] \to M)_{n \in \mathbb{N}}$ between*

*$x$ and $y$ with* $\lim_n \int_0^{t_n} L(t, \sigma_n(t), \dot{\sigma}_n(t))\, dt = 0$. *Then*

$$\sup_{t \in [0, t_n]} \|\dot{\sigma}_n(t) - X_t(\sigma_n(t))\| \underset{n\to\infty}{\to} 0$$

The lemma shows that these curves $\sigma_n$ get closer to the set $\{(t, x, v) \mid v = X_t(x)\}$ as $n$ grows. This enables to deduce that the Mather set (and the Aubry set) are contained in the zero-level of $L$.

*Proof.* Arguing by contradiction, suppose that there exists $\delta > 0$ such that we can find an increasing sequence $k_n$ of integers verifying for all $n \in \mathbb{N}$, $\sup_{t\in[0,t_{k_n}]} \|\dot{\sigma}_{k_n}(t) - X_t(\sigma_{k_n}(t))\| > \delta$.

Let $n \in \mathbb{N}$. There exists $s_n \in [0, t_{k_n}]$ such that $\|\dot{\sigma}_{k_n}(s_n) - X_{s_n}(\sigma_{k_n}(s_n))\| > \delta$. Thus $(s_n, \sigma_{k_n}(s_n), \dot{\sigma}_{k_n}(s_n))$ belongs to the open set $U$ of $\mathbb{T}^1 \times TM$ defined as

$$U := \left\{ (t, x, v) \in \mathbb{T}^1 \times TM \;\middle|\; L(t, x, v) > \frac{\delta^2}{2} \right\} \tag{8.1.17}$$

We also define the set $F$ as

$$F := \left\{ (t, x, v) \in \mathbb{T}^1 \times TM \mid L(t, x, v) = 0 \right\} \tag{8.1.18}$$

It is easy to see that $F$ and its complement are invariant under the map $\Phi_L^\tau$ defined in (5.2.5). And knowing that $\bar{U} \subset F^c$, we get the inclusion

$$\Phi_L^{[0,1]}(\bar{U}) := \{(t+\tau, \phi_L^{t,t+\tau}(x, v) \mid (t, x, v) \in \bar{U},\ \tau \in [0, 1]\} \subset F^c$$

Since $L$ is continuous and since the sets $\Phi_L^{[0,1]}(\bar{U})$ and $F$ are disjoint with one of them being closed and the other compact in $\mathbb{T}^1 \times TM$, there exists $\nu > 0$ such that

$$\Phi_L^{[0,1]}(\bar{U}) \subset \left\{ (t, x, v) \in \mathbb{T}^1 \times TM \mid L(t, x, v) > \nu \right\} \tag{8.1.19}$$

Now, we study the action along any of the curves $\sigma_n$

$$A_L(\sigma_n) = \int_0^{t_n} L(t, \sigma_n(t), \dot{\sigma}_n(t))\, dt = \int_0^{t_n} \frac{1}{2} \|\dot{\sigma}_n(t) - X_t(\sigma_n(t))\|^2\, dt \underset{n\to\infty}{\longrightarrow} 0$$

Thus there exists $N > 0$ such that for all $n \geq N$, $A_L(\sigma_n) < \nu$.

Let $n \in \mathbb{N}$ be such that $k_n > N$, and suppose that $s_n < t_{k_n} - 1$, otherwise use $\Phi_L^{[-1,0]}(\bar{U})$ instead of $\Phi_L^{[0,1]}(\bar{U})$.

$$A_L(\sigma_{k_n}) = \int_0^{t_{k_n}} L(t, \sigma_{k_n}(t), \dot{\sigma}_{k_n}(t))\, dt \geq \int_{s_n}^{s_n+1} L(t, \sigma_{k_n}(t), \dot{\sigma}_{k_n}(t))\, dt$$

However, since $\sigma_{k_n}$ is a minimizing curve, the variational theory claims that it follows

Euler-Lagrange flow as mentioned in Theorem 5.1.9. This implies that for all $t \in [s_n, s_n+1]$,

$$(t, \sigma_{k_n}(t), \dot\sigma_{k_n}(t)) = \Phi_L^{t-s_n}(s_n, \sigma_{k_n}(s_n), \dot\sigma_{k_n}(s_n)) \in \Phi_L^{[0,1]}(\bar U)$$

Therefore we get from the inclusion (8.1.19) that

$$A_L(\sigma_{k_n}) \geq \int_{s_n}^{s_n+1} L(t, \sigma_{k_n}(t), \dot\sigma_{k_n}(t))\, dt \geq \nu$$

which contradicts the definition of the definition of $N$. □

*Proof of Proposition 8.1.12.* Fix $\varepsilon > 0$. Let $\delta > 0$ be a real number. Using the lemma, we get a minimizing cruve $\sigma : [0, T] \to M$ from $x$ to $y$ with $T > \tau$ such that for all $t \in [0, T]$, $\|\dot\sigma(t) - X_t(\sigma(t))\| \leq \delta$.
Let $\alpha \in \mathbb{N}$ and $0 \leq \beta < \tau$ be such that $T = \alpha\tau + \beta$. For $k \in \{1, .., \alpha - 1\}$, We define the curves $\gamma_0 : [0, \tau + \beta] \to M$ and $\gamma_k : [k\tau + \beta, (k+1)\tau + \beta] \to M$ as follows

$$\begin{cases} \dot\gamma_k(t) = X_t(\gamma_k(t)) \\ \gamma_k(S_k) = \sigma(S_k) \end{cases} \tag{8.1.20}$$

where we denoted by $[S_k, T_k]$ the domains of $\gamma_k$. Thus we have for all $t \in [S_k, T_k]$,

$$\begin{aligned} \|\sigma(t) - \gamma_k(t)\| &= \left\| \int_{S_k}^t \dot\sigma(s) - X_s(\gamma(s))\, ds \right\| \\ &\leq \int_{S_k}^t \|\dot\sigma(s) - X_s(\sigma(s))\|\, ds + \int_{S_k}^t \|X_s(\sigma(s)) - X_s(\gamma(s))\|\, ds \\ &\leq (T_k - S_k)\delta + \int_{S_k}^t M.\|\sigma(s) - \gamma_k(s)\|\, ds \\ &\leq 2\tau\delta + \int_{S_k}^t M.\|\sigma(s) - \gamma_k(s)\|\, ds \end{aligned}$$

where $M$ is a Lipschitz constant for the vector field $X_t$.
And using the Grönwall lemma, we get the upper bound

$$\forall t \in [S_k, T_k], \quad \|\sigma(t) - \gamma_k(t)\| \leq 2\tau\delta.e^{M(t-S_k)} \leq 2\tau\delta.e^{M2\tau}$$

Taking $\delta < \varepsilon.\frac{e^{-M2\tau}}{2\tau}$ gives the desired estimations. □

A specialization of the previous proposition to the $(n,k)$-Peierls Barriers $h^{n\infty+k}$ (see Definition 7.4.3) leads to the variation below

**Corollary 8.1.14.** *Let $n \in \mathbb{N}^*$ and $0 \leq k \leq n-1$. Let $x$ and $y$ be two points of $M$ such that $h^{n\infty+k}(x,y) = 0$. Then for every $\varepsilon > 0$, there exists a finite family of curves $\big(\gamma_i : [0, T_i] \to M\big)_{0 \leq i \leq m}$ with integer times $T_i$ such that*

*i.* $\gamma_0(0) = x$.

*ii.* $T_0 = k$ *and for all* $i > 0$ $T_i$ *is a multiple of* $n$.

*iii.* *For all* $0 \le i \le m$, $\gamma_i(t) = f_t(\gamma_i(0))$.

*iv.* *For all* $0 \le i \le m-1$, $d\big(\gamma_i(T_i), \gamma_{i+1}(0)\big) < \varepsilon$, *and* $d\big(\gamma_m(T_m), y\big) < \varepsilon$.

### Quantitative Properties of the Peierls Barriers $h^{k\infty}$

We now present a lemma that compiles essential properties needed for our analysis, particularly in the study of the non-wandering set $\Omega(\mathcal{T})$ and in facilitating the selection of a smooth element within it.

We define the integer-time $\alpha$-limit set $\alpha_k(\gamma)$ and the integer-time $\omega$-limit set $\omega_k(\gamma)$ of a curve $\gamma : \mathbb{R} \to M$ as

$$\begin{aligned} \alpha_k(\gamma) &= \{z \in M \mid \exists (q_n)_n \in \mathbb{N}^{\mathbb{N}};\ \lim_n q_n = +\infty,\ \lim_n \gamma(-k.q_n) = z\} \\ \omega_k(\gamma) &= \{z \in M \mid \exists (q_n)_n \in \mathbb{N}^{\mathbb{N}};\ \lim_n q_n = +\infty,\ \lim_n \gamma(k.q_n) = z\} \end{aligned} \tag{8.1.21}$$

Recall from (8.1.12) the vector fields $Y_t$ and $Z$ used to define $X_t$.

**Lemma 8.1.15.** *1. Fix an integer* $k \ge 1$. *Let* $x$ *be a point of* $M$ *and let* $y$ *and* $z$ *be two respective points of* $\alpha_k(f_t(x))$ *and* $\omega_k(f_t(x))$. *Then*

$$h^{k\infty}(y, x) = h^{k\infty}(x, z) = h^{k\infty}(y, z) = 0 \tag{8.1.22}$$

*2. Let* $x$ *be a* 1*-periodic point under the flow* $f_t$. *Then for all integer* $k \ge 1$ *and* $i \ge 0$, $h^{k\infty+i}(x, \cdot) = h^{\infty}(x, \cdot)$ *and* $h^{k\infty+i}(\cdot, x) = h^{\infty}(\cdot, x)$.

*3. Let* $F$ *be an arc-wise connected subset of* $M$ *such that the vector fields* $X_t$ *and* $Y_t$ *coincide on* $f_t(F)$ *and that there exist an integer* $m$ *such that* $\mathcal{R}_m(F) = F$. *Then, for all pair of points* $x$ *and* $y$ *in* $F$, *we have*

$$\overline{h}(x, y) := \limsup_{k \to \infty} h^k(x, y) = 0 \tag{8.1.23}$$

*In particular, for all integers* $k \ge 1$ *and* $i \ge 0$,

*i.* $h^{k\infty+i}(x, y) = 0$

*ii.* *For any point* $z \in M$, $h^{k\infty+i}(x, z) = h^{k\infty+i}(y, z)$ *and* $h^{k\infty+i}(z, x) = h^{k\infty+i}(z, y)$. *We will sometimes denote these quantities respectively by* $h^{k\infty+i}(F, z)$ *and* $h^{k\infty+i}(z, F)$.

*4. Let* $F$ *be a subset defined as above and assume moreover that it is a closed* $f_t$*-invariant subset of* $M$. *Assume that the rotation* $f_t = \mathcal{R}_t$ *is* $k$*-periodic on* $F$. *Then, for all pair of points* $x$ *and* $y$ *in* $M \setminus F$ *which are separated by* $F$, *we have*

$$h^{k\infty}(x, y) = h^{k\infty}(x, F) + h^{k\infty}(F, y) \tag{8.1.24}$$

*Proof.* 1. This is an immediate implication of the Liminf property (7.3.5) of the Peierls Barrier. More precisely,

$$0 \le h^{k\infty}(y,x) \le \liminf_i A_L(f_t(x)_{|[-ki,0]}) = 0$$

The other cases are analogous.

2. Let $x$ be a 1-periodic point under the flow $f_t$. We fix two integers $k \ge 1$ and $i \ge 0$. We already know by definition of the Peierls barriers that

$$h^\infty(x,y) = \liminf_n h^n(x,y) \le \liminf_n h^{kn+i}(x,y) \le h^{k\infty+i}(x,y)$$

Now let $\gamma_n : [0,k_n] \to M$ be a sequence of curves linking $x$ to $y$ with increasing integer times $k_n$ and such that $h^\infty(x,y) = \lim_n A_L(\gamma_n)$. We left-concatenate the curves $\gamma_n$ with the loops $f_t(x) : [0,(k-1).k_n + i] \to M$ based on the 1-time periodic point $x$. We obtain new curves $\tilde{\gamma}_n : [0, k.k_n + i] \to M$ still linking $x$ to $y$. Since $f_t$ is of null action by the Mañé Lagrangian $L$, we get

$$h^\infty(x,y) = \lim_n A_L(\gamma_n) = \lim_n A_L(\tilde{\gamma}_n) \ge \liminf_n h^{kn+i}(x,y) = h^{k\infty+i}(x,y)$$

The equality follows. The other equality is analogous.

3. Fix two points $x$ and $y$ in $F$. We distinguish two cases. The case where $y$ is periodic under the rotation $\mathcal{R}_t$ and the case where $\mathcal{R}_t$ is an irrational rotation at $y$.

First assume that $y$ is periodic under $\mathcal{R}_t$ with integer period $\rho \ge 1$. We can consider for all integer $0 \le i < \rho$, the points $y_i = \mathcal{R}_i^{-1}(y)$ and the curves $\gamma^i : [0,1] \to F$ connecting $x$ to $y_i$. The latter exist thanks to the arc-wise connectedness of $F$. Now for all integer time $k \ge 1$, we define the curve $\gamma_k : [0,k] \to M$ by

$$\gamma_k(t) = \mathcal{R}_t \circ \gamma^i\left(\frac{t}{k}\right)$$

where $0 \le i < \rho$ is such that $k \equiv i \pmod{\rho}$. Then, its velocity is given by

$$\dot{\gamma}_k(t) = \frac{d\mathcal{R}_t}{dt} \circ \gamma^i\left(\frac{t}{k}\right) + d\mathcal{R}_t.\frac{\dot{\gamma}^i}{k}\left(\frac{t}{k}\right) = Y_t(\gamma_k(t)) + \frac{1}{k} d\mathcal{R}_t.\dot{\gamma}^i\left(\frac{t}{k}\right)$$

Using the fact that $X_t = Y_t$ on $f_t(F) = \mathcal{R}_t(F)$, we get

$$\begin{aligned} A_L(\gamma_k) &= \int_0^k L(\tau, \gamma_k(\tau), \dot{\gamma}_k(\tau))\, d\tau = \int_0^k \frac{1}{2}\|\dot{\gamma}_k(\tau) - Y_\tau(\gamma_k(\tau))\|^2 d\tau \\ &= \int_0^k \frac{1}{2k^2}\left\|d\mathcal{R}_\tau.\dot{\gamma}^i\left(\frac{\tau}{k}\right)\right\|^2 d\tau \le \|d\mathcal{R}_t\|_\infty^2 . \max_{0\le j<\rho} \|\dot{\gamma}^j\|_\infty^2 . \frac{1}{2k} \to 0 \quad \text{as } k \to \infty \end{aligned}$$

Therefore, we deduce that

$$0 \le \overline{h}(x,y) = \limsup_{k\to\infty} h^k(x,y) \le \limsup_{k\to\infty} A_L(\gamma_k) = 0 \tag{8.1.25}$$

Now assume that $\mathcal{R}_t$ is an irrational rotation at $y$. Fix $\varepsilon > 0$ and consider a large $n$ such that the points $y_i = \mathcal{R}_i^{-1}(y)$, $0 \le i < n$ are $\varepsilon$-dense in the orbit of $y$. For all integer $k \ge 1$ and $0 \le i < n$, we consider curves $\gamma^i : [0, 1-\varepsilon/k] \to F$ linking $x$ to $y_i$ with a uniform $C^1$-bound over $k$. Now fix $k \ge 1$ and let $0 \le i < n$ be such that $d(y_k, y_i) < \varepsilon$ where $y_k = \mathcal{R}_k^{-1}(y)$. Consider a curve $\tilde{\gamma}^i : [0,\varepsilon] \to M$ linking $y_i$ to $y_k$ and such that $\|\dot{\tilde{\gamma}}^i\|_\infty \le \varepsilon$. Now define the curve $\gamma_k : [0,k] \to M$ by

$$\gamma_k(t) = \begin{cases} \mathcal{R}_t \circ \gamma^i\left(\frac{t}{k}\right) & \text{if } t \in [0, k-\varepsilon] \\ \tilde{\gamma}^i(t-k+\varepsilon) & \text{if } t \in [k-\varepsilon, k] \end{cases}$$

Then, doing the same computations as in the periodic case, we obtain

$$\begin{aligned} A_L(\gamma_k) &= \int_0^{k-\varepsilon} L(\tau, \gamma_k(\tau), \dot{\gamma}_k(\tau))\, d\tau + \int_{k-\varepsilon}^{k} L(\tau, \gamma_k(\tau), \dot{\gamma}_k(\tau))\, d\tau \\ &\le \int_0^{k-\varepsilon} \frac{1}{2k^2} \left\| d\mathcal{R}_\tau . \dot{\gamma}^i\left(\frac{\tau}{k}\right)\right\|^2 d\tau + \int_{k-\varepsilon}^{k} L(\tau, \gamma_k(\tau), \dot{\gamma}_k(\tau))\, d\tau \\ &\le \|d\mathcal{R}_t\|_\infty^2 . \max_{0\le j<n} \|\dot{\gamma}^j\|_\infty^2 . \frac{1}{2k} + \left( \sup_{\substack{(\tau,z)\in\mathbb{T}^1\times M \\ \|v\|\le\varepsilon}} L(\tau, z, v) \right) . \varepsilon \end{aligned}$$

Taking $k \ge \frac{1}{\varepsilon}$, we conclude that $\lim_k A_L(\gamma_k) = 0$ and that $\overline{h}(x,y) = 0$.

The remaining claimed identities follow from

$$0 \le h^{k\infty+i}(x,y) \le \overline{h}(x,y) = 0$$

and from the triangular inequality (5.2.28) of Proposition 5.2.14 applied twice as below

$$\begin{aligned} h^{k\infty+i}(z,x) &\le h^{k\infty+i}(z,y) + h^{k\infty}(y,x) = h^{k\infty+i}(z,y) \\ &\le h^{k\infty+i}(z,x) + h^{k\infty}(x,y) = h^{k\infty+i}(z,x) \end{aligned}$$

4. Let $x$ and $y$ be two points of $M \setminus F$ separated by $F$. We know from the triangular inequality (5.2.28) that

$$h^{k\infty}(x,y) \le h^{k\infty}(x,F) + h^{k\infty}(F,y) \tag{8.1.26}$$

We need to prove the inverse inequality. Let $\gamma_i : [0, k.n_i] \to M$ be curves linking $x$ to $y$ such that $h^{k\infty}(x,y) = \lim_i A_L(\gamma_i)$. Since $F$ separates $x$ and $y$, there exist times $t_i \in (0, k.n_i)$ such that the points $z_i := \gamma_i(t_i)$ belong to $F$.

We concatenate the curve $\gamma_{i|[0,t_i]}$ with the curve $f_{t_i,t}(z_i) : \left[t_i, k\lceil \frac{t_i}{k}\rceil\right] \to M$ to get a first curve $\gamma_i^1 : \left[0, k\lceil \frac{t_i}{k}\rceil\right] \to M$ linking $x$ to $z_i^1 := f_{t_i,k\lceil \frac{t_i}{k}\rceil}(z_i)$. And since the flow $f_t$ of $X_t$ is of null action, we have $A_L(\gamma_{i|[0,t_i]}) = A_L(\gamma_i^1)$.

Similarly, we concatenate the curve $f_{t_i,t}(z_i) : \left[k\lfloor \frac{t_i}{k}\rfloor, t_i\right] \to M$ with the curve $\gamma_{i|[t_i,kn_i]}$ to get a curve $\gamma_i^2 : \left[k\lfloor \frac{t_i}{k}\rfloor, kn_i\right] \to M$ linking $z_i^2 := f_{t_i,k\lfloor \frac{t_i}{k}\rfloor}$ to $y$. And we still have $A_L(\gamma_{i|[t_i,kn_i]}) = A_L(\gamma_i^2)$.

For $j = 1,2$, the points $z_i^j$ do belong to the $f_t$-invariant set $F$. And by compactness, we can assume that they converge up to extraction to $z^j \in F$. We get

$$\begin{aligned} h^{k\infty}(x,y) &= \lim_i A_L(\gamma_i) = \lim_i A_L(\gamma_i^1) + A_L(\gamma_i^2) \\ &\geq h^{k\infty}(x,z^1) + h^{k\infty}(z^2,y) \\ &= h^{k\infty}(x,F) + h^{k\infty}(F,y) \end{aligned}$$

where we used the Liminf property (7.3.5) in the second line. This gives the wanted inequality. □

**Remark 8.1.16.** These lemmas will be applied in various ways in the different sets (8.1.1) that served the construction of $f_t$.

The last two properties of the lemma can be applied to the connected components of $D$ or to the set $B_n \setminus \bigcup_i B_n^i$ which satisfy all the assumptions on $F$.

As an example, we present an evaluation of $h^{\rho_n\infty}(x_n^i, D)$ in dimension $d \geq 3$. This quantity is well defined due to the Point 3 of the lemma and the fact that $D$ is connected in high dimensions. Since $F = \partial B_n$ separates $x_n^i$ and $D$, we have

$$h^{\rho_n\infty}(x_n^i, D) = h^{\rho_n\infty}(x_n^i, \partial B_n) + h^{\rho_n\infty}(\partial B_n, D)$$

The dynamics of $f_t$ in $A_n$ is such that for all $x \in A_n$, $\alpha_{\rho_n}(f_t(x)) \subset \partial B_n$ and $\omega_{\rho_n}(f_t(x)) \subset \partial C_n \subset D$ so that Property 1 implies $h^{\rho_n\infty}(\partial B_n, D) = 0$. Thus, we get the equality

$$h^{\rho_n\infty}(x_n^i, D) = h^{\rho_n\infty}(x_n^i, \partial B_n)$$

## 8.2 Construction of a Non-Periodic Recurrent Viscosity Solution

We now proceed with constructing a non-wandering viscosity solution that is not periodic, thus partially proving Theorem 8.0.1. We will select a solution $u(t,x)$ such that $u(t,x_n)$ is periodic with a minimal period $\rho_n$. And since $\rho_n$ diverges to infinity, $u(t,x)$

cannot be periodic. However, the regularity conditions will not yet be met, and a suitable choice of a smooth solution will be deferred to Section 8.3.

Additionally, we will examine the dynamics of $\mathcal{T}$ when restricted to the $\omega$-limit set $\omega(u)$, and more broadly to $\omega(v)$ for any non-wandering viscosity solution $v \in \Omega(\mathcal{T})$. We will observe that $\Omega(u)$ forms a Cantor set within $\mathcal{C}(M,\mathbb{R})$, where $\mathcal{T}$ behaves as an odometer. And in the general case, we will see through a proof of Theorem 8.0.2 and Proposition 8.2.32 that $\mathcal{T}_{|\omega(v)}$ is a factor of an odometer.

### 8.2.1 Choice of the Non-Periodic Recurrent Initial Data

Now that the framework has been established, our objective is to identify a suitable scalar map $u$ with a non-periodic $\omega$-limit set $\omega(u)$ under the action of the Lax-Oleinik semi-group $\mathcal{T}$. To achieve this, we start by constructing a $\rho_n$-periodic viscosity solution at every $\rho_n$-orbit $\{x_n^i\}$ of the Lagrangian flow $\phi_L$.

Recall from Proposition 7.3.2 that the barriers $h^{\rho_n\infty}(x_n^0,\cdot)$ are $\rho_n$-periodic viscosity solutions. Consequently, a fitting candidate for an initial data $u : M \to \mathbb{R}$ would be

$$u(x) = \inf_{n\geq 0}\{h^{\rho_n\infty}(x_n,x)\} \tag{8.2.1}$$

where $x_n = x_n^0$ were defined in (8.1.2).

**Theorem 8.2.1.** *The viscosity solution with initial data $u$ defined in (8.2.1) is a recurrent, non-periodic viscosity solution of the Hamilton-Jacobi equation (5.1.8).*

*Proof.* The proof is segmented into two main parts. One for non-periodicity and another for recurrence. However, before delving into these steps, we need to determine the action of the Lax-Oleinik semigroup $\mathcal{T}$ on the initial data $u$ of (8.2.1).

#### Evaluation of $\mathcal{T}^k u$

We evaluate the action of the Lax-Oleinik operator $\mathcal{T}$ on the chosen scalar map $u$.

**Proposition 8.2.2.** *For all integer $k \geq 0$, we have*

$$\mathcal{T}^k u(x) = \inf_{n\geq 0}\{h^{\rho_n\infty+k}(x_n,x)\} \tag{8.2.2}$$

*Proof.* We know from Proposition 7.3.2 that the maps $h^{\rho_n\infty}(\cdot,x_n,\cdot)$ are viscosity solutions.

Hence, Proposition 6.2.1 immediately yields

$$\mathcal{T}^k u(x) = \inf_{n\geq 0}\{\mathcal{T}^k h^{\rho_n\infty}(x_n, x)\} = \inf_{n\geq 0}\{h^{\rho_n\infty+k}(x_n, x)\}$$

□

In order to determine the exact periodicity of $u$ around every orbit $\{x_n^i\}$, we need to take a closer look on the behaviour of the $\rho_n$-barrier $h^{\rho_n\infty}$ for the studied Mañé Lagrangian $L$.

**Proposition 8.2.3.** *For all integers $n \geq 0$, the map $h^{\rho_n\infty}(x_n,\cdot)$ is a periodic viscosity solution with minimal period $\rho_n$.*

*Proof.* The Proposition 7.3.2 already tells that the maps $h^{\rho_n\infty}(x_n,\cdot)$ are $\rho_n$-periodic viscosity solutions of the Hamilton-Jacobi equation (5.1.8). However, the minimality of the periods $\rho_n$ requires extra effort. We claim that

(i) $h^{\rho_n\infty}(x_n, x_n) = 0$.

(ii) For all $k \in \{1, .., \rho_n - 1\}$, $\mathcal{T}^k h^{\rho_n\infty}(x_n, x_n) = h^{\rho_n\infty+k}(x_n, x_n) \neq 0$.

**(i)** For this case, One needs to follow the flow $f_t$ starting at $x_n$. Let $\gamma : \mathbb{R} \to M$ be the curve $\gamma(t) = f_t(x_n)$ which, by construction of the point $x_n$, is $\rho_n$-periodic. We have by definition of the $\rho_n$-barrier $h^{\rho_n\infty}$ that

$$\begin{aligned} h^{\rho_n\infty}(x_n, x_n) = \liminf_k h^{\rho_n k}(x_n, x_n) \leq \liminf_k A_L(\gamma_{|[0,\rho_n k]}) &= \liminf_k \int_0^{\rho_n k} L(\tau, \gamma(\tau), \dot\gamma(\tau))\, d\tau \\ &= \liminf_k \int_0^{\rho_n k} \frac{1}{2}\|\dot\gamma(\tau) - X_\tau(\gamma(\tau))\|^2 d\tau \\ &= \liminf_k \int_0^{\rho_n k} \frac{1}{2}\|\partial_\tau f_\tau(x_n) - X_\tau(f_\tau(x_n))\|^2 d\tau = 0 \end{aligned}$$

where the last nullity is due to the fact that $f_t$ is the flow of $X_t$. Combining this inequality with the non-negativity of the Mañé Lagrangian barriers noted in Corollary 8.1.9, we deduce that $h^{\rho_n\infty}(x_n, x_n) = 0$.

**(ii)** Let $k \in \{1, .., \rho_n - 1\}$. It suffices to see that the kind of chain transitivity claimed by Corollary 8.1.14 does not hold in this case. We act by contradiction and suppose that $h^{\rho_n\infty+k}(x_n, x_n) = 0$.

Recall that the ball $B_n^k$ centered at $x_n^k$ with radius $\delta_n$ is the basin of attraction of $x_n^k$ under the map $f_{\rho_n}$. Fix a radius $0 < \delta < \frac{\delta_n}{2}$. We know that $f_{\rho_n}$ sends the open ball $B_\delta = B(x_n^k, \delta)$ compactly into itself so that $f_{\rho_n}(\overline{B_\delta}) \subset B_\delta$. Let $\varepsilon$ be defined as $\varepsilon = d\big(f_{\rho_n}(\overline{B_\delta}), \partial B_\delta\big)$. Then we have $\varepsilon < \delta$ and $f_{\rho_n}(B_\delta) \subset B_{\delta-\varepsilon}$.

Since we assumed $h^{\rho_n\infty+k}(x_n, x_n) = 0$, we can now apply the Corollary 8.1.14 to this $\varepsilon$. We use the same notations for the obtained curves. Given that $\gamma_0(t) = f_t(x_n)$, it follows that $\gamma_0(k) = x_n^k$. This enables to localize $\gamma_1(0)$ as below

$$\gamma_1(0) \in B(\gamma_0(k), \varepsilon) = B(x_n^k, \varepsilon) \subset B(x_n^k, \delta) = B_\delta$$

Using the definition of $\varepsilon$ and the fact $T_1$ is a multiple of $\rho_n$, we deduce that $\gamma_1(T_1)$ belongs to $f_{\rho_n}(B_\delta) \subset B_{\delta-\varepsilon}$. And as a result, the localization of the next initial point $\gamma_2(0)$ gives $\gamma_2(0) \in B_\delta$.

A simple induction leads to the inclusion $x_n \in B_\delta \subset B_n^k = B_{\delta_n}$ which contradicts the fact the $x_n^k$-centred ball $B_n^k$ and the $x_n$-centred ball $B_n^0$ are disjoint. □

**Remark 8.2.4.** Note that the proof of point (ii) yields the following result : For every integers $n > 0$ and $0 \leq k < \rho_n$, and every point $x \neq x_n^k$, we have $h^{\rho_n\infty+k}(x_n, x) > 0$.

This proof provides the necessary details to verify the hypothesis of the contrapositive version of Proposition 8.1.12, which establishes the strict positivity of $\liminf_{t\to 0} h^t(x, y) > 0$ or $\liminf_{n\to 0} h^{t_n}(x, y) > 0$.

From now on, we will omit such detailed explanations and only assert the non-existence of pseudo-orbits linking two studied point based on dynamics of the flow $f_t$ between these.

### Non-Periodicity of $u$.

We proceed with the proof of the non-periodicity of $u$. Arguing by contradiction, suppose that there exists a positive integer $q > 0$ such that $\mathcal{T}^q u = u$. Given that the sequence of periods $\rho_n$ diverges to infinity, we can fix an integer $n \geq 0$ such that $\rho_n > q$. We claim that

(i) For $k = 0, \rho_n$, $\mathcal{T}^k u(x_n) = 0$.

(ii) For $k = 1, .., \rho_n - 1$, $\mathcal{T}^k u(x_n) \neq 0$.

which contradicts the $q$-periodicity of $(\mathcal{T}^k u(x_n))_{k\geq 0}$ with $q < \rho_n$.

The point (i) is a consequence of Proposition 8.2.2 combined with the proof of Proposition 8.2.3. When gathered, we get for $k = 0, \rho_n$

$$0 \leq \mathcal{T}^k u(x_n) \leq h^{\rho_n\infty+k}(x_n, x_n) = h^{\rho_n\infty}(x_n, x_n) = 0$$

The second point (ii) is more subtle. Fix an integer $1 \leq k < \rho_n$. We know from the proof of Proposition 8.2.3 that $h^{\rho_n\infty+k}(x_n, x_n) > 0$. However, we need a uniform positive lower bound on $h^{\rho_m\infty+k}(x_m, x_n)$ for $m \neq n$ in order to deduce that $\mathcal{T}^k u(x_n) = \inf_{m\geq 0}\{h^{\rho_m\infty+k}(x_m, x_n)\} > 0$.

Fix an integer $m \neq n$. We use the dynamics of the flow $f_t$. We know from the construction that the flow is directed from $\partial B_n$ towards $\partial C_n$ (see Figure 8.1). We will use this fact to apply the contrapositive part of Proposition 8.1.12.

But before that, let us take a deeper look on the minimizing curves that realize $h^{\rho_m\infty+k}(x_m, x_n)$. There exists a sequence of minimizing curves $\gamma_i : [0, \rho_m q_i + k] \to M$ from $x_m$ to $x_n$ such that $h^{\rho_m\infty+k}(x_m, x_n) = \lim_i A_L(\gamma_i)$. Since $\partial C_n$ and $\partial B_n$ separate successively $x_m$ from $x_n$, there exist two real times $0 \leq s_i < t_i \leq \rho_n q_i + k$ such that $\gamma_i(s_i) \in \partial C_n$ and $\gamma_i(t_i) \in \partial B_n$. However, we know from the à priori compactness Theorem 9.5.11 that there is a uniform Lipschitz value $M > 0$ for all minimizing curves. Hence, we have

$$d(\partial C_n, \partial B_n) \leq d(\gamma_i(s_i), \gamma_i(t_i)) \leq M.(t_i - s_i)$$

Set $\tau = \frac{d(\partial C_n, \partial B_n)}{M}$ which is independent of $m$. The following holds.

**Lemma 8.2.5.** *There exists a real number $\varepsilon_n > 0$ depending only on $\tau$ such that*

$$\inf\{h^{s,t}(x,y) \mid 0 \leq s < t,\ t - s \geq \tau,\ x \in \partial C_n,\ y \in \partial B_n\} \geq \varepsilon_n \tag{8.2.3}$$

*Proof.* We argue by contradiction. Assume that there exists a sequence of curves $\sigma_i : [s_i, t_i] \to M$ such that $t_i - s_i \geq \tau$, $x_i := \sigma_i(s_i) \in \partial C_n$, $y_i := \sigma_i(t_i) \in \partial B_n$ and $h^{s_i,t_i}(x_i, y_i) = A_L(\sigma_i) \to 0$ as $i \to +\infty$. The time periodicity of the Lagrangian allows, up to time translation, to assume that $s_i \in \mathbb{T}^1$ for all $i \geq 0$. And by compactness of $\mathbb{T}^1 \times \partial C_n \times \partial B_n$,we can further assume that, up to extraction, $(s_i, x_i, y_i)$ converges to $(s, x, y) \in \mathbb{T}^1 \times \partial C_n \times \partial B_n$.

If the times $t_i$ are bounded, we can also assume that $t_i$ converges to $t \geq s + \tau$. The continuity of the potential $h$ results in the equality

$$h^{s,t}(x,y) = \lim_i h^{s_i,t_i}(x_i, y_i) = 0$$

This gives rise to a minimizing curve $\gamma : [s, t] \to M$ linking $x$ to $y$ such that

$$A_L(\gamma) = \int_s^t \frac{1}{2} \|\dot\gamma(\zeta) - X_\zeta(\gamma(\zeta)\|^2 d\zeta = 0 \tag{8.2.4}$$

Thus, $\dot\gamma(\zeta) = X_\zeta(\gamma(\zeta))$ and the curve follows the flow $f_t$. However, we know from the construction that the flow $f_t$ is directed from $\partial B_n$ toward $\partial C_n$, which contradicts the existence of such a curve $\gamma$.

We showed that the times $t_i$ are unbounded, diverging to $+\infty$. Using the lipschitz-regularity of the potentials $h$ stated in Proposition 5.1.13, We have for all integer $i \geq 0$

$$|h^{s,t_i}(x,y) - h^{s_i,t_i}(x_i, y_i)| \leq \kappa_\tau.d\big((s,x,y),(s_i,x_i,y_i)\big) \to 0 \quad \text{as } i \to \infty$$

Hence, we get $\liminf_i h^{s,t_i}(x,y) = 0$, which contradicts the contrapositive claim of Proposition 8.1.12. □

The lemma above established that $A_L(\gamma_{i|[s_i,t_i]}) \geq \varepsilon_n$, providing the uniform lower bound

$$h^{\rho_m\infty+k}(x_m,x_n) = \lim_i A_L(\gamma_i) \geq \liminf_i A_L(\gamma_{i|[s_i,t_i]}) \geq \varepsilon_n$$

Therefore

$$\mathcal{T}^k(u)(x_n) \geq \inf_{m\geq 0}\{h^{\rho_m\infty+k}(x_m,x_n)\} \geq \min\{\varepsilon_n, h^{\rho_n\infty+k}(x_n,x_n)\} > 0 \tag{8.2.5}$$

This concludes the proof of point (ii) and the proof of the non-periodicity of $u$.

**Remark 8.2.6.** The proof for point (ii) could have been simplified by exploiting the autonomous nature of the radial flow $g_t$ from $\partial B_n$ to $\partial C_n$. We made a deliberate choice not to take advantage of this feature.

### Recurrence of $u$

Set for all $n \geq 0$, the integer $p_n = \prod_{k=0}^{n} \rho_k$. The $\rho_n$-periodicity of the $\rho_n$-barriers $h^{\rho_n\infty}$ tells that

$$\begin{aligned}\mathcal{T}^{p_k}u(x) &= \inf_{n\geq 0}\{h^{\rho_n\infty+p_k}(x_n,x)\}\\ &= \min\Big\{\inf_{0\leq n\leq k}\{h^{\rho_n\infty}(x_n,x)\}, \inf_{n>k}\{h^{\rho_n\infty+p_k}(x_n,x)\}\Big\}\end{aligned} \tag{8.2.6}$$

We see that the $k$ first elements of the infimum defining $\mathcal{T}^{p_k}u$ coincide with those in the infimum defining $u$. Our expectation is that the difference between these two maps diminishes as $k$ approaches infinity.

We need to compare the two maps $h^{\rho_n\infty}(x_n,x)$ and $h^{\rho_n\infty+i}(x_n,x)$ for some $0 \leq i < \rho_n$. A lemma in this direction is the following

**Lemma 8.2.7.** *For all integers $n \geq 0$ and $0 \leq i < \rho_n$,*

$$h^{\rho_n\infty+i}(x_n,x) = h^{\rho_n\infty}(x_n^i,x) \tag{8.2.7}$$

*Proof.* We know that $h^i(x_n,x_n^i) = 0$ and $h^{\rho_n-i}(x_n^i,x_n) = 0$ as they are respectively realized by the curves $f_t(x_n)$ and $f_t(x_n^i)$. Hence, applying the triangular inequality (5.2.10), we obtain

$$h^{\rho_n\infty+i}(x_n,x) = \liminf_j h^{\rho_n j+i}(x_n,x)$$
$$\leq \liminf_j \left[h^i(x_n,x_n^i) + h^{\rho_n j}(x_n^i,x)\right] = \liminf_j h^{\rho_n j}(x_n^i,x) = h^{\rho_n\infty}(x_n^i,x)$$

and

$$h^{\rho_n\infty}(x_n^i,x) = h^{\rho_n\infty+\rho_n}(x_n^i,x) = \liminf_j h^{\rho_n j+\rho_n}(x_n^i,x)$$
$$\leq \liminf_j \left[h^{\rho_n-i}(x_n^i,x_n) + h^{\rho_n j+i}(x_n,x)\right] = \liminf_j h^{\rho_n j+i}(x_n,x) = h^{\rho_n\infty+i}(x_n,x)$$

□

We proceed to the comparison. Recall from the definition of the $n$-Peierls barriers and from Proposition 7.3.2 that $h^{\rho_n\infty}$ is $\kappa_1$-Lipschitz. Thus, we get for all integer $n>k$ and all point $x$ in $M$,

$$|h^{\rho_n\infty+p_k}(x_n,x) - h^{\rho_n\infty}(x_n,x)| = |h^{\rho_n\infty}(x_n^{p_k},x) - h^{\rho_n\infty}(x_n,x)|$$
$$\leq 2\kappa_1.d(x_n^{p_k},x_n) \leq 2\kappa_1.2r_n \leq 4\kappa_1.r_k =: \varepsilon_k$$

Since we have equality $h^{\rho_n\infty+p_k} = h^{\rho_n\infty}$ for all $n\leq k$, it follows that for all integer $n\geq 0$ and all $x\in M$

$$|h^{\rho_n\infty+p_k}(x_n,x) - h^{\rho_n\infty}(x_n,x)| \leq \varepsilon_k \tag{8.2.8}$$

In particular,

$$h^{\rho_n\infty}(x_n,x) - \varepsilon_k \leq h^{\rho_n\infty+p_k}(x_n,x) \leq h^{\rho_n\infty}(x_n,x) + \varepsilon_k$$

Taking the infimum over $n$ and using (8.2.6), we get

$$u(x) - \varepsilon_k \leq \mathcal{T}^{p_k}u(x) \leq u(x) + \varepsilon_k$$

or more precisely

$$\|\mathcal{T}^{p_k}u(x) - u(x)\|_\infty \leq \varepsilon_k \to 0 \quad \text{as } k\to+\infty$$

This means that $\mathcal{T}^{p_k}u$ converges to $u$ in $\mathcal{C}(M,\mathbb{R})$, indicating that $u$ belongs to its own $\omega$-limit set $\omega(u)$ and is a recurrent viscosity solution. □

**Remark 8.2.8.** Some remarks are to be made on this result.

1. As mentioned in the introduction, this Theorem 8.2.1 implies that, at the difference of non-autonomous [Fat98] or the one-dimensional case [BR04], there is no convergence of the Lax-Oleinik operator to a periodic orbit.

2. Recall from Proposition 6.3.3 and Remark 6.3.4 the Minimality of the Lax-Oleinik operator $\mathcal{T}$ on $\omega(u)$. This means that all the elements $v$ of $\omega(u)$ are non-periodic, recurrent viscosity solutions. A deeper study of the $\omega$-limit set of the constructed $u$ will be done in the next Subsection 8.2.2.

3. The arguments of the proof do work for a large variety of examples. It seems uncomplicated to construct various Hamiltonians that admit non-periodic recurrent viscosity solutions. An example where the Mather set contains a Cantor set would be as follows. $f = \tau \circ g$ where $g$ is represented in black arrows in the Figure 8.2 below and $\tau$ is the map that exchanges the interior components of the lemniscates and acts as an adding machine (See Subsection 8.2.2 for a definition) on the Cantor set formed by the intersection of these different components. The latter map is represented in pink in Figure 8.2.

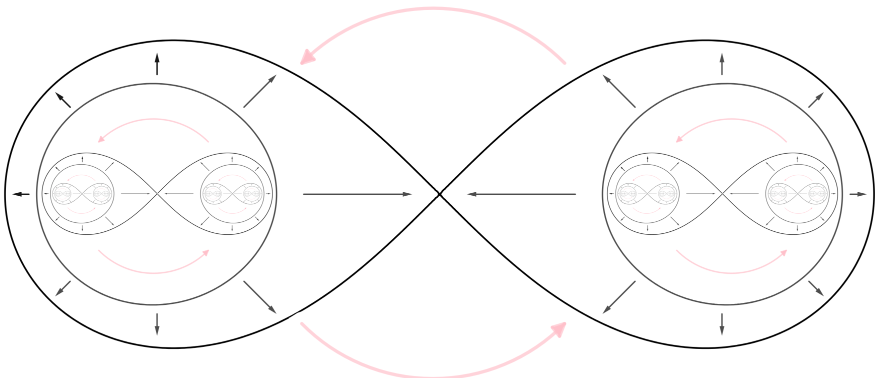


FIGURE 8.2 – Dynamics of $f$.

### 8.2.2 Action of the Lax-Oleinik Operator $\mathcal{T}$ on the Non-Wandering Set $\Omega(\mathcal{T})$

In this subsection, we outline the construction of all recurrent viscosity solutions associated with the Mané Lagrangian $L$ defined in (8.1.14). Furthermore, we provide insights into the dynamics of $\mathcal{T}$ within their $\omega$-limit sets.

To begin, we focus on examining the $\omega$-limit set $\omega(u)$ of the previously established recurrent viscosity solution $u$ defined in (8.2.1). There are slight topological differences between the cases where the manifold $M$ is of dimension $d = 2$ or $d \geq 3$. And since the proofs are analogous, we will address only the high-dimensional case, noting the differences with the 2D case in the remarks.

### The $\omega$-Limit Set $\omega(u)$ and Adding Machines

First and foremost, it is essential to establish a means of identifying each element within the $\omega$-limit set $\omega(u)$. To do so, we start by giving a convenient expression of $\mathcal{T}^k u$.

**Proposition 8.2.9.** *If $d \geq 0$, For all integers $k \in \mathbb{N}$, we have*

$$\mathcal{T}^k u(x) = \begin{cases} \min\left(h^\infty(z_\infty, x), h^{\rho_n \infty}(x_n^k, x)\right) & \textit{if } x \in C_n \\ 0 & \textit{if } x \in \overline{D} \end{cases}$$

*Proof.* Fix an integer $n \in \mathbb{N}$. Let $u_n^k$ be the map defined by

$$u_n^k(x) = \inf_{m \neq n} \{h^{\rho_m \infty}(x_m^k, x)\} \tag{8.2.9}$$

We know from Proposition 8.2.2 and Lemma 8.2.7 that

$$\begin{aligned} \mathcal{T}^k u(x) &= \inf_{m \geq 0} \{h^{\rho_m \infty + k}(x_m, x)\} = \inf_{m \geq 0} \{h^{\rho_m \infty}(x_m^k, x)\} \\ &= \min\left(u_n^k(x), h^{\rho_n \infty}(x_n^k, x)\right) \end{aligned} \tag{8.2.10}$$

We show that for all $x \in C_n$, $u_n^k(x) = h^\infty(z_\infty, x)$. Let $x$ be a fixed point of $C_n$ and fix an integer $m \neq n$. Let us evaluate $h^{\rho_m \infty}(x_m^k, x)$. The set $\overline{D'}$ separates $x$ and $x_m^k$. Then, applying Property 4 of Lemma 8.1.15 applied to $F = \overline{D'}$ followed by an application Property 2 to the 1-periodic point $z_\infty$, yields

$$\begin{aligned} h^{\rho_m \infty}(x_m^k, x) &= h^{\rho_m \infty}(x_m^k, D') + h^{\rho_m \infty}(D', x) \\ &= h^{\rho_m \infty}(x_m^k, z_\infty) + h^{\rho_m \infty}(z_\infty, x) \\ &= h^{\rho_m \infty}(x_m^k, z_\infty) + h^\infty(z_\infty, x) \end{aligned} \tag{8.2.11}$$

Moreover, the regularity of the barriers, as stated in Proposition 7.3.2, added with the fact that $h^{\rho_m \infty}(x_m^k, x_m^k) = 0$, provide

$$0 \leq h^{\rho_m \infty}(x_m^k, z_\infty) = h^{\rho_m \infty}(x_m^k, z_\infty) - h^{\rho_m \infty}(x_m^k, x_m^k) \leq \kappa_1 . d(z_\infty, x_m^k) \leq \kappa_1 . r_m \to 0 \quad \text{as } m \to \infty$$

Thus, $\inf_{m \neq n}\{h^{\rho_m \infty}(x_m^k, z_\infty)\} = 0$ and using (8.2.11), we obtain

$$\begin{aligned} u_n^k(x) &= \inf_{m \neq n} \{h^{\rho_m \infty}(x_m^k, x)\} \\ &= \inf_{m \neq n} \{h^{\rho_m \infty}(x_m^k, z_\infty) + h^\infty(z_\infty, x)\} \\ &= \inf_{m \neq n} \{h^{\rho_m \infty}(x_m^k, z_\infty)\} + h^\infty(z_\infty, x) \\ &= h^\infty(z_\infty, x) \end{aligned}$$

Furthermore, the Property 3 of Lemma 8.1.15 shows that for all $x \in \overline{D}$ and all integer

$m$, $h^{\rho_m\infty}(x_m^k, x) = h^{\rho_m\infty}(x_m^k, z_\infty)$ which, in the same fashion as above, yields for all $x \in \overline{D}$, $u_n^k(x) = h^\infty(z_\infty, x) = 0$. We showed that

$$u_n^k(x) = \begin{cases} h^\infty(z_\infty, x) & \text{if } x \in C_n \\ 0 & \text{if } x \in \overline{D} \end{cases}$$

This, added to identity (8.2.10), concludes the proof. □

**Remark 8.2.10.** In the two dimensional case, the proposition doesn't hold. However, by the same arguments, we still have for all $x \in C_n$ that $u_n^k(x) = u_n^0(x)$, and if we consider the sets

$$\begin{aligned} D_0' &:= \{r \geq r_1 + 2\delta_1\} \supset C_0 \\ D_n' &:= \{r_{n+1} + 2\delta_{n+1} \leq r \leq r_{n-1} - 2\delta_{n-1}\} \supset C_n \quad \text{if } n \geq 1 \end{aligned} \tag{8.2.12}$$

then

$$\mathcal{T}^k u(x) = \min\left(u_n^0(x), h^{\rho_n\infty}(x_n^k, x)\right) \quad \text{if } x \in D_n' \tag{8.2.13}$$

The subsequent proposition allows the desired identification.

**Proposition 8.2.11.** *Every element $v$ of $\omega(u)$ is characterized by a unique infinite sequence $\underline{k}(v) = (k_n(v))_{n\geq 0}$ in $Z_{\underline{\rho}} := \prod_{n=0}^\infty \mathbb{Z}/\rho_n\mathbb{Z}$ such that*

$$v(x) = \inf_{n\geq 0}\{h^{\rho_n\infty + k_n(v)}(x_n, x)\} \tag{8.2.14}$$

*Proof.* Proposition 8.2.9 show that

$$\begin{aligned} \mathcal{T}^k u(x) &= \inf_{m\geq 0}\{h^{\rho_m\infty + k}(x_m, x)\} \\ &= \begin{cases} \min\left(h^\infty(z_\infty, x), h^{\rho_n\infty}(x_n^k, x)\right) & \text{if } x \in C_n \\ 0 & \text{if } x \in D \end{cases} \end{aligned}$$

Given that the finiteness of the sets $\{x_n^k\}_{k\geq 0}$, it follows that for any $v \in \omega(u)$ and for all $n \geq 0$, there must exist an integer $k_n(v) \in \{0, 1, \ldots, \rho_n - 1\}$ such that

$$v(x) = \begin{cases} \min\left(h^\infty(z_\infty, x), h^{\rho_n\infty}(x_n^{k_n(v)}, x)\right) & \text{if } x \in C_n \\ 0 & \text{if } x \in D \end{cases} \tag{8.2.15}$$

Let $\tilde{v} : M \to \mathbb{R}$ be the map defined by

$$\tilde{v}(x) = \inf_{n\geq 0}\{h^{\rho_n\infty + k_n(v)}(x_n, x)\} \tag{8.2.16}$$

The exact same proof as for Proposition 8.2.9 applied to $\tilde{v}$ shows that

$$\begin{aligned}\tilde{v}(x) &= \inf_{n\geq 0}\{h^{\rho_n\infty + k_n(v)}(x_n, x)\}\\ &= \begin{cases}\min\left(h^\infty(z_\infty, x), h^{\rho_n\infty}(x_n^{k_n(v)}, x)\right) & \text{if } x\in C_n\\ 0 & \text{if } x\in D\end{cases}\\ &= v(x)\end{aligned}$$

*Unicity of $k_n(v)$.* An application of the quantitative Property 1 of Lemma 8.1.15 gives for all $i\in\{0,..,\rho_n-1\}$,

$$h^\infty(z_\infty, x_n^i) = h^\infty(z_\infty, y_n) > 0 \quad \text{and} \quad h^{\rho_n\infty}(x_n^{k_n(v)}, x_n^i) = \begin{cases} h^\infty(x_n^{k_n(v)}, y_n) > 0 & \text{if } i\neq k_n(v)\\ 0 & \text{if } i = k_n(v)\end{cases}$$

This implies that $v(x_n^i) = 0$ if and only if $i = k_n(v)$. In other words, for all integer $n\geq 0$, $k_n(v)$ is the index $i$ of the unique point $x_n^i$ such that $v(x_n^i) = 0$. This results in the uniqueness of $k_n(x)$. □

**Remark 8.2.12.**
1. In the proof, it is evident that each element $v$ in $\omega(u)$ is uniquely determined by the images of the points $\{x_n^i\ ;\ 0\leq i<\rho_n, n\geq 0\}\subset\mathcal{M}_0$. This follows from a broader uniqueness theorem proved in proposition 3.2 of [BR04], which asserts that all recurrent viscosity solutions are uniquely characterized by their images on the projected Mather set $\mathcal{M}_0$.
2. The identity (8.2.13) suffices to prove the proposition in the 2D case.

We have shifted the analysis of $\omega(u)$ from the space $\mathcal{C}(M,\mathbb{R})$ to $Z_{\underline{\rho}} = \prod_{n=0}^\infty \mathbb{Z}/\rho_n\mathbb{Z}$. This enables to compare the dynamics of $\mathcal{T}_{|\omega(u)}$ to other dynamical systems on the new space $Z_{\underline{\rho}}$.

Let us define a topology in the space $Z_{\underline{\rho}} = \prod_{n=0}^\infty \mathbb{Z}/\rho_n\mathbb{Z}$ by endowing it with a metric $d$ defined as follows

$$d\left(\underline{q},\underline{p}\right) = \frac{1}{2^{\nu(\underline{q},\underline{p})}} \quad \text{with} \quad \nu\left(\underline{q},\underline{p}\right) = \min\{n\geq 0 \mid q_n\neq p_n\} \tag{8.2.17}$$

**Definition 8.2.13.** The *odometer map* also called the *adding machine* on $Z_{\underline{\rho}} = \prod_{n=0}^\infty \mathbb{Z}/\rho_n\mathbb{Z}$ is the homeomorphism $\tau$ defined by

$$\begin{array}{rccc}\tau: & \prod_{n=0}^\infty \mathbb{Z}/\rho_n\mathbb{Z} & \longrightarrow & \prod_{n=0}^\infty \mathbb{Z}/\rho_n\mathbb{Z}\\ & \underline{q} = (q_2, q_3,\ldots) & \longmapsto & \underline{q}+\underline{1} = (q_2+1, q_3+1,\ldots)\end{array} \tag{8.2.18}$$

**Remark 8.2.14.** For more on adding machines, a standard reference is [HR79]. A survey from a dynamical perspective is available in [Dow05]. And for a recent and concise introduction, we refer to [HV22].

1. In the standard definition, adding machines are minimal maps on their domains which is not necessarily the case of the map $\tau$. However, if $\underline{q}$ is a non-periodic point of $\tau$, then the restriction of $\tau$ to the $\omega$-limit set $\omega\left(\underline{q}\right)$ behaves as an adding machine in the usual sense.

   In this chapter, we chose to include periodic maps in what we refer as odometers or adding machines.

2. Note that the odometer $\tau$ is an isometry of the space $Z_{\underline{\rho}}$. Indeed, if the first $n$-terms of the sequences $\underline{q}$ and $\underline{p}$ coincide, the same holds for $\underline{q}+1$ and $\underline{p}+1$ so that $d(\tau(\underline{q}),\tau(\underline{p})) = d(\underline{q},\underline{p})$.

We now state the theorem that completes the understanding of the asymptotic behaviour of the constructed recurrent viscosity solution $u$ introduced in (8.2.1).

**Theorem 8.2.15.** *The restriction of the Lax-Oleinik semigroup $\mathcal{T}$ to the $\omega$-limit set $\omega(u)$ is a factor of the odometer map $\tau$. More precisely, there exists a continuous injective map $\varphi : \omega(u) \hookrightarrow Z_{\underline{\rho}}$ such that the following commutative diagram holds*

$$\begin{array}{ccc} \omega(u) & \xrightarrow{\mathcal{T}} & \omega(u) \\ \varphi\downarrow & & \downarrow\varphi \\ Z_{\underline{\rho}} & \xrightarrow{\tau} & Z_{\underline{\rho}} \end{array} \tag{8.2.19}$$

*Proof.* The choice of the map $\varphi$ comes from the unicity Proposition 8.2.11. For all $v$ in $\omega(u)$, we set $\varphi(v) = \underline{k}(v)$. Then the injectivity of the map $\varphi$ follows immediately. We only need to prove that it is continuous.

For all $n \geq 0$, we set $v_n^i : M \to \mathbb{R}$ to be the map

$$v_n^i(x) = \min\left(h^{\infty}(z_{\infty}, x), h^{\rho_n\infty}(x_n^i, x)\right)$$

and the constants $\lambda_n$ to be defined as

$$\lambda_n = \min_{0\leq i\neq j<\rho_n}\left\{\left\|v^i_{n|C_n} - v^j_{n|C_n}\right\|_{\infty}\right\}$$

We have $\lambda_n > 0$. Indeed, if we take an integer $0 \leq i < \rho_n$, then we know from Proposition 8.1.12 that

$$v_n^i(x_n^i) = h^{\rho_n\infty}(x_n^i, x_n^i) = 0\,, \qquad h^{\infty}(z_{\infty}, x_n^i) > 0\,, \quad \text{and} \quad \forall j\neq i,\ h^{\rho_n\infty}(x_n^j, x_n^i) > 0$$

This yields $\left\|v^i_{n|C_n} - v^j_{n|C_n}\right\|_{\infty} > 0$, and taking the infimum on $i \neq j$, we obtain the positivity $\lambda_n > 0$.

We can show that the sequence $\lambda_n$ is decreasing. But instead of proving it, we consider the positive non-increasing sequence of $\tilde{\lambda}_n$ defined by

$$\tilde{\lambda}_n = \min_{2 \le i \le n} \lambda_i$$

Let $v$ and $w$ be two elements of $\omega(u)$ such that $\|v - w\|_\infty < \tilde{\lambda}_n$. We know from Proposition 8.2.11 that for all $n \ge 0$,

$$v_{|C_n} = v_n^{k_n(v)} \quad \text{and} \quad w_{|C_n} = v_n^{k_n(w)}$$

Hence, from the definition of $\tilde{\lambda}_n$ and the hypothesis $\|v - w\|_\infty < \tilde{\lambda}_n$, we deduce that for all $2 \le i \le n$, $k_i(v) = k_i(w)$ which means that

$$d(\varphi(v), \varphi(w)) \le \frac{1}{2^{n+1}}$$

This proves the (uniform) continuity of $\varphi$ and concludes the proof of the theorem. □

**Corollary 8.2.16.** *The $\omega$-limit set $\omega(u)$ is a Cantor space.*

*Proof.* The topology induced by the metric $d$ on the space $\prod_{n=0}^{\infty} \mathbb{Z}/\rho_n\mathbb{Z}$ is the cylinder topology which is totally disconnected. The Theorem 8.2.15 asserts that $\omega(u)$ is embedded in $\prod_{n=0}^{\infty} \mathbb{Z}/\rho_n\mathbb{Z}$. Thus, it is totally disconnected. Moreover, we know that $\omega(u)$ is a compact metric space and Proposition 6.3.3 and Remark 6.3.4 indicate that it has no periodic points and hence no isolated points since every element is neared by its own orbit. These properties collectively imply that $\omega(u)$ is homeomorphic to the Cantor set. □

It is possible to provide a precise description of the minimal odometer $\tau : \varphi(\omega(u)) \to \varphi(\omega(u))$. The classification of these (minimal) odometers is a well-known subject, and we will make use of a classification theorem to be stated below. But first, let us introduce a more classical definition of odometers.

**Definition 8.2.17.** For all sequence of integers $\underline{k} = (k_n)_{n \ge 0}$, we define the *$\underline{k}$-odometer* $\tau_{\underline{k}} : Z_{\underline{k}} \to Z_{\underline{k}}$, defined on the space $Z_{\underline{k}} = \prod_{n=0}^{\infty} \mathbb{Z}/k_n\mathbb{Z}$ endowed with the metric (8.2.17), as follows

$$\tau_{\underline{k}}(\underline{q}) = \begin{cases} (\underbrace{0, 0, \cdots, 0}_{n \text{ times}}, q_n + 1, q_{n+1}, q_{n+2}, \dots) & \text{if } n = \min\{m \ge 0\,;\, q_m \ne k_m - 1\} < +\infty \\ \underline{0} = (0, 0, 0, \dots) & \text{if } n = +\infty \end{cases} \tag{8.2.20}$$

**Remark 8.2.18.** These are the classical maps that we refer to as odometers. Examining their behaviour, it becomes clearer to understand why they are named adding machines. As mentioned in Remark 8.2.14, they are transitive and even minimal. And there is an odometer for every positive sequence $\underline{k}$.

**Definition 8.2.19.** For all sequence of integers $\underline{k} = (k_n)_{n\geq 0}$, we define its *multiplicity function* $m_{\underline{k}}$ by

$$\begin{array}{rccc} m_{\underline{k}}: & \mathbb{P} & \longrightarrow & \mathbb{N}\cup\{\infty\} \\ & p & \longmapsto & \sum_{n\geq 0}\nu_p(k_n) \end{array} \tag{8.2.21}$$

where $\mathbb{P}$ is the set of prime numbers and $\nu_p$ is the $p$-adic valuation $\nu_p(k) = \max\{i \in \mathbb{N} \mid p^i \text{ divides } k\}$.

**Theorem 8.2.20.** *(Classification of adding machines [BS95] [HV22]) For any two positive sequences of integers $\underline{k}$ and $\underline{q}$, the classical odometers $\tau_{\underline{k}}$ and $\tau_{\underline{q}}$ are topologically conjugate if and only if the sequences $\underline{k}$ and $\underline{q}$ have the same multiplicity function $m_{\underline{k}} = m_{\underline{q}}$.*

**Corollary 8.2.21.** *Let $m : \mathbb{P}\to\mathbb{N}\cup\{\infty\}$ be the map defined by*

$$m(p) = \sup\{\nu_p(\rho_n)\ ;\ n\geq 0\} \tag{8.2.22}$$

*Then the odometer $\mathcal{T}_{|\omega(u)}$ is topologically conjugate to any $\underline{\rho}'$-odometer $\tau_{\underline{\rho}'}$ with multiplicity function $m_{\underline{\rho}'} = m$.*

*Proof.* The first step is to understand the $\tau$-orbit of $\underline{0}$ in $Z_{\underline{\rho}}$ which has been proven in Theorem 8.2.15 to be homeomorphic to the $\mathcal{T}$-orbit of $u$ in $\mathcal{C}(M,\mathbb{R})$. For all integer $n\geq 0$, consider the following clopen $n$-cylinder

$$\begin{aligned} C(\underline{0},n) &= \left\{\underline{k}\in Z_{\underline{\rho}}\ \middle|\ d(\underline{0},\underline{k}) \leq \frac{1}{2^{n+1}}\right\} \\ &= \left\{\underline{k}\in Z_{\underline{\rho}} \mid k_i = 0,\ i = 0,..,n\right\} \end{aligned} \tag{8.2.23}$$

As noticed in the second point of Remark 8.2.14, the odometer $\tau$ is an isometry on $Z_{\underline{\rho}}$ so that for all integer $k\in\mathbb{Z}$,

$$\tau^k(C(\underline{0},n)) = C(\tau^k(\underline{0}),n)$$

We would like to associate each of these cylinders to the $(n+1)$-tuples of $Z_{\underline{\rho},n} = \prod_{i=0}^n \mathbb{Z}/\rho_i\mathbb{Z}$ formed by the first $n$ terms of the sequences $\tau^k(\underline{0})$. For that purpose, we define the set $C_{\underline{\rho},n}$ of $(n+1)$-cylinders of $Z_{\underline{\rho}}$ and the conjugacy map

$$\begin{array}{rccc} \psi_{\underline{\rho},n}: & C_{\underline{\rho},n} & \longrightarrow & Z_{\underline{\rho},n} \\ & C(\underline{q},n) & \longmapsto & \underline{q}_n = (q_0,\dots,q_n) \end{array} \tag{8.2.24}$$

This conjugacy allows to identify the cylinders to $(n+1)$-tuples of $Z_{\underline{\rho},n}$.

If we denote by $\mathrm{Orb}(\tau, C(\underline{0},n))$ the $\tau$-orbit of the cylinder $C(\underline{0},n)$, we get the following diagram

$$\begin{tikzcd} \mathrm{Orb}(\tau,C(\underline{0},n)) \arrow[r, "\tau"] \arrow[d, "\psi_{\underline{\rho},n}"] & \mathrm{Orb}(\tau,C(\underline{0},n)) \arrow[d, "\psi_{\underline{\rho},n}"] \\ Z_{\underline{\rho},n} \arrow[r, "\tau_n"] & Z_{\underline{\rho},n} \end{tikzcd} \tag{8.2.25}$$

where $\tau_n$ is the odometer $\tau_n(\underline{q}) = \underline{q} + \underline{1}$ on $Z_{\underline{\rho},n}$. As a result, the $\tau$-orbit of $C(\underline{0},n)$ corresponds to the $\tau_n$-orbit of $\underline{0}_n = (0,..,0)$ in $Z_{\underline{\rho},n}$.

*Understanding* $\mathrm{Orb}(\tau_n,\underline{0}_n)$. The set $\mathrm{Orb}(\tau_n,\underline{0}_n)$ corresponds to the subgroup of the additive group $(\prod_{i=0}^n \mathbb{Z}/\rho_i\mathbb{Z},+)$ generated by $\underline{1}_n = (1,..,1)$. Moreover, the theory of finite abelian groups shows that the order of this element $\underline{1}_n$ in $\prod_{i=0}^n \mathbb{Z}/\rho_i\mathbb{Z}$ is equal to the exponent of the entire group, which is $\mathrm{lcm}(\rho_i\ ;\ i=0\,..\,n)$ where lcm stands for least common multiple.

*Construction of a* $\underline{\rho}'$*-odometer* $\tau_{\underline{\rho}'}$. We would like to rewrite the diagram (8.2.25) with isometric columns onto a new space $Z_{\underline{\rho}',n} = \prod_{i=0}^n \mathbb{Z}/\rho_i'\mathbb{Z}$ endowed with its usual metric (8.2.17). We also aim to replace $\tau_n$ with a map $\tau_{\underline{\rho}',n}$ derived from the action of a minimal $\underline{\rho}'$-odometer $\tau_{\underline{\rho}'}$ on the cylinder $C(\underline{0},n)$ of $Z_{\underline{\rho}'}$ as in the diagram (8.2.25). To achieve this, note that by the discussion above, we have determined the precise cardinality of each orbit $\mathrm{Orb}(\tau, C(\underline{0},n))$. Building on this, we set for all integer $n \geq 1$

$$\rho_0' = \rho_0 \quad \text{and} \quad \rho_n' = \frac{\mathrm{lcm}(\rho_i\ ;\ i=0\,..\,n)}{\mathrm{lcm}(\rho_i\ ;\ i=0\,..\,n-1)} \tag{8.2.26}$$

and consider the space $Z_{\underline{\rho}'} = \prod_{n=0}^\infty \mathbb{Z}/\rho_n'\mathbb{Z}$ and the corresponding $\underline{\rho}'$-odometer $\tau_{\underline{\rho}'}$. If we denote by $C'(\underline{0}',n)$ the $n$-cylinder around $\underline{0}' = (0,0,0,...)$ in the $Z_{\underline{\rho}'}$, then by definition of the minimal $\underline{\rho}'$-odometer $\tau_{\underline{\rho}'}$, we have

$$\mathrm{Orb}(\tau_{\underline{\rho}'}, C'(\underline{0}',n)) \simeq Z_{\underline{\rho}',n} := \prod_{i=0}^n \mathbb{Z}/\rho_i'\mathbb{Z}$$

and since

$$\#Z_{\underline{\rho}',n} = \prod_{i=0}^n \rho_i' = \rho_0 . \prod_{i=1}^n \frac{\mathrm{lcm}(\rho_j\ ;\ j=0\,..\,i)}{\mathrm{lcm}(\rho_j\ ;\ j=0\,..\,i-1)} = \mathrm{lcm}(\rho_i\ ;\ i=0\,..\,n) = \#\,\mathrm{Orb}(\tau, C(\underline{0},n))$$

This allows to define, up to conjugacy by the map $\psi_{\underline{\rho}',n}$ analogous to (8.2.24), a bijective map $\varphi_n : \mathrm{Orb}(\tau, C(\underline{0},n)) \to \mathrm{Orb}(\tau_{\underline{\rho}'}, C'(\underline{0}',n))$ as

$$\varphi_n(\tau_n^k(\underline{0}_n)) = \tau_{\underline{\rho}',n}^k(\underline{0}_n') \tag{8.2.27}$$

Thus, we get the following diagram

$$\begin{tikzcd} \mathrm{Orb}(\tau, C(\underline{0},n)) \arrow[r, "\tau_n"] \arrow[d, "\varphi_n"] & \mathrm{Orb}(\tau, C(\underline{0},n)) \arrow[d, "\varphi_n"] \\ Z_{\underline{\rho}',n} \simeq \mathrm{Orb}(\tau_{\underline{\rho}'}, C'(\underline{0}',n)) \arrow[r, "\tau_{\underline{\rho}',n}"] & Z_{\underline{\rho}',n} \simeq \mathrm{Orb}(\tau_{\underline{\rho}'}, C'(\underline{0}',n)) \end{tikzcd} \tag{8.2.28}$$

where the columns are bijections.

*Construction of the conjugacy between $\tau_{|\omega(\underline{0})}$ and $\tau_{\underline{\rho}'}$.* Let us define the bijective map $\varphi : \mathrm{Orb}(\tau, \underline{0}) \to \mathrm{Orb}(\tau_{\underline{\rho}'}, \underline{0}')$ by

$$\varphi(\tau^k(\underline{0})) = \tau_{\underline{\rho}'}^k(\underline{0}') \tag{8.2.29}$$

which is well defined since the two orbits are infinite. We show that it is an isometry. Let $\underline{q}$ and $\underline{p}$ be two elements of $\mathrm{Orb}(\tau, \underline{0})$ and let $\underline{q}' = \varphi(\underline{q})$ and $\underline{p}' = \varphi(\underline{p})$ be their respective images. We have

$$d(\underline{q}', \underline{p}') = \frac{1}{2^{\nu(\underline{q}', \underline{p}')}}$$

with

$$\begin{aligned} \nu\left(\underline{q}', \underline{p}'\right) &= \min\{n \geq 0 \mid q_n' \neq p_n'\} \\ &= \min\{n \geq 0 \mid \underline{q}_n' \notin C'(\underline{p}_n', n)\} \\ &= \min\left\{n \geq 0 \mid \underline{q}_n' \notin \varphi_n\big(C(\underline{p}_n, n)\big)\right\} \end{aligned}$$

where in the second line we used the definition (8.2.23) of the cylinders $C'(\underline{p}_n', n)$. If we consider two integers $k_q$ and $k_p$ such that $\underline{q} = \tau^{k_q}(\underline{0})$ and $\underline{p} = \tau^{k_p}(\underline{0})$. Then, we deduce from the diagram (8.2.28) that

$$\underline{q}_n' = \tau_{\underline{\rho}',n}^{k_q}(\underline{0}_n) \notin \varphi_n\big(C(\tau_{\underline{\rho}',n}^{k_p}(\underline{0}_n), n)\big) \quad \text{if and only if} \quad \underline{q}_n = \tau_n^{k_q}(\underline{0}_n) \notin C(\tau_n^{k_p}(\underline{0}_n), n)$$

Thus, we get

$$\nu\left(\underline{q}', \underline{p}'\right) = \min\{n \geq 0 \mid \underline{q}_n \notin C(\underline{p}_n, n)\} = \nu\left(\underline{q}, \underline{p}\right)$$

and

$$d(\underline{q}', \underline{p}') = \frac{1}{2^{\nu(\underline{q}', \underline{p}')}} = \frac{1}{2^{\nu(\underline{q}, \underline{p})}} = d(\underline{q}, \underline{p})$$

Therefore, $\varphi : \mathrm{Orb}(\tau, \underline{0}) \to \mathrm{Orb}(\tau_{\underline{\rho}'}, \underline{0}') \subset Z_{\underline{\rho}'}$ is an isometry. In particular, it is a uniformly continuous isometry from $\mathrm{Orb}(\tau, \underline{0})$ to a subset of the complete compact space $Z_{\underline{\rho}'}$. Consequently,, it is possible to extend it to the closure $\omega(\underline{0})$ of $\mathrm{Orb}(\tau, \underline{0})$, providing an isometric injection $\varphi : \omega(\underline{0}) \to Z_{\underline{\rho}'}$. Additionally, we know from the minimality of the $\underline{\rho}'$-odometer $\tau_{\underline{\rho}'}$ that $\omega(\underline{0}') = Z_{\underline{\rho}'}$. Thus, the conjugacy relation (8.2.29) and the continuity of $\varphi$ induce the inclusion

$$Z_{\underline{\rho}'} = \omega(\underline{0}') \subset \varphi(\omega(\underline{0}))$$

This implies that the map $\varphi : \omega(\underline{0}) \to Z_{\underline{\rho}'}$ is a bijective isometry. Furthermore, the continuity of the odometers $\tau$ and $\tau_{\underline{\rho}'}$ allow to extend the conjugacy relation to $\omega(\underline{0})$, resulting in the desired commutative diagram

$$\begin{tikzcd} \omega(\underline{0}) \arrow[r, "\tau"] \arrow[d, "\varphi"] & \omega(\underline{0}) \arrow[d, "\varphi"] \\ Z_{\underline{\rho}'} \arrow[r, "\tau_{\underline{\rho}'}"] & Z_{\underline{\rho}'} \end{tikzcd} \tag{8.2.30}$$

where the columns are bijective isometries.

*Conclusion.* We showed that the odometer $\tau_{|\omega(u)}$ is topologically conjugate to the $\underline{\rho}'$-odometer $\tau_{\underline{\rho}'}$ for $\underline{\rho}'$ defined in (8.2.26). Let us compute its multiplicity map. For any prime number $p$, we have

$$\begin{aligned} m_{\underline{\rho}'}(p) &= \sum_{n=0}^{\infty} \nu_p(\rho'_n) = \nu_p(\rho_0) + \sum_{n=1}^{\infty} \nu_p(\operatorname{lcm}(\rho_i \; ; \; i = 0 \,..\, n)) - \nu_p(\operatorname{lcm}(\rho_i \; ; \; i = 0 \,..\, n-1)) \\ &= \lim_n \nu_p(\operatorname{lcm}(\rho_i \; ; \; i = 0 \,..\, n)) = \lim_n \sup_{2 \le i \le n} \nu_p(\rho_i) = \sup_{n \ge 0} \nu_p(\rho_n) = m(p) \end{aligned}$$

We have shown that the sequence $\underline{\rho}'$ has as a multiplicity function the map $m$ mentioned in the statement. An application of the classification Theorem 8.2.20 concludes the proof of the corollary. □

### Description of the Non-Wandering Set $\Omega(\mathcal{T})$

In this section, we extend our analysis beyond the $\omega$-limit of the specific viscosity solution $u$. We will describe the $\omega$-limits of any recurrent viscosity solution $v$ associated with the constructed Lagrangian $L$ defined in (8.1.14). Our approach involves applying a generalized representation formula on $\Omega(\mathcal{T})$ following Chapter 7.

But first, we need to identify the Mather set $\mathcal{M}_0$ expressed in the following proposition.

**Proposition 8.2.22.** *For the Mañé Lagrangian $L$ defined in (8.1.14), the time-zero projected Mather set $\mathcal{M}_0$ is given by*

$$\begin{aligned} \mathcal{M}_0 = \mathcal{A}_0 &= M \smallsetminus \bigcup_{n \ge 0} \left( A_n \cup \bigcup_{0 \le i < \rho_n} \left( B_n^i \smallsetminus \{x_n^i\} \right) \right) \\ &= \overline{D} \cup \bigcup_{n \ge 0} \left( \overline{B_n} \smallsetminus \bigcup_{0 \le i < \rho_n} B_n^i \right) \cup \bigcup_{\substack{n \ge 0 \\ 0 \le i < \rho_n}} \{x_n^i\} \end{aligned} \tag{8.2.31}$$

*and the projected Mather set $\mathcal{M}$ is given by*

$$\mathcal{M} = \{(t, \mathcal{R}_t(x)) \mid x \in \mathcal{M}_0, t \in \mathbb{R}\} \tag{8.2.32}$$

*Proof.* We know from the inclusion $\mathcal{M}_0 \subset \mathcal{A}_0$ of Proposition 5.2.16 that a point $x$ of

$\mathcal{M}_0$ verifies $h^\infty(x,x) = 0$. Thus, we can drop all the points that have a positive Peierls Barrier $h^\infty(x,x) > 0$. According to Proposition 8.1.12, all non-chain-recurrent points by the flow $f_t$ are to be dropped. This eliminated set $E$ of non-chain-recurrent points, equals by construction to

$$E := \bigcup_{n=0}^{\infty}\left(A_n \cup \bigcup_{0\le i<\rho_n}\left(B_n^i \smallsetminus \{x_n^i\}\right)\right)$$

We deduce the inclusion

$$\begin{aligned}\mathcal{M}_0 \subset M \smallsetminus E &= \overline{D} \cup \bigcup_{n\ge 0}\left(\overline{B_n} \smallsetminus \bigcup_{0\le i<\rho_n} B_n^i\right) \cup \bigcup_{\substack{n\ge 0\\ 0\le i<\rho_n}} \{x_n^i\}\\ &= \overline{D'} \cup \bigcup_{n\ge 0}\overline{D_n} \bigcup_{n\ge 0}\left(\overline{B_n} \smallsetminus \bigcup_{0\le i<\rho_n} B_n^i\right) \cup \bigcup_{\substack{n\ge 0\\ 0\le i<\rho_n}} \{x_n^i\}\end{aligned}$$

Let us verify the inverse inclusion. For any point $x$ of $\overline{D'}$, and for $v = X_0(x) = X_t(x) = 0$, the measure $\mu = dt \otimes \delta_{(x,0)}$, where $\delta_{(x,0)}$ is the Dirac at $(x,v)$, is an invariant probability measure with

$$\int_{\mathbb{T}^1\times TM} L\, d\mu = \int_0^1 \frac{1}{2}\|v - X_t(x)\|^2\, dt = 0 = -\alpha_0$$

Hence, $\mu$ is a minimizing measure and $x$ belongs to $\pi(\mathrm{Supp}(\mu)) \cap \{t=0\} \subset \mathcal{M}_0$.

Let $x$ be one of the $x_n^i$ or be a point of $\overline{B_n} \smallsetminus \bigcup_{0\le i<\rho_n} B_n^i$. The point $x$ is $\rho_n$-periodic by the flow $f_t$ and so is $(x,v) = (x, X_0(x))$ by the Lagrangian flow $\phi_L^t$. Let $\gamma : \mathbb{R} \to M$ be the $\rho_n$-periodic loop, projection on $M$ of the loop $\phi_L^t(x,v) = \big(\gamma(t), X_t(\gamma(t))\big)$. And let $\mu$ be the uniform measure on the graph of $(t,\gamma(t)) \in \mathbb{T}^1 \times TM$. Once again, $\mu$ is a minimizing measure with support $\mathrm{Supp}(\mu)$ being equal to the graph of $(t,\gamma(t))$. Hence, $x = \pi(\gamma(0))$ belongs to $\mathcal{M}_0$.

Now let $x$ be a point of $\overline{D_n}$. If it is periodic by $X_t(x) = Y_t(x)$, then it is in the Mather set $\mathcal{M}_0$ by the preceding case. The rotation number of the orbit $\mathcal{R}_t(x)$ is $\frac{1}{\rho_n}\eta_n(x)$ which only depends on $d(x,O_n)$. And since $\eta_n(x)$ is continuous and increasing with $d(x,O_n)$, we obtain a dense set $Q$ in $[2\delta_n, 3\delta_n]$ such that $\eta_n(Q) \subset \mathbb{Q}$. The set $\{x \in M \mid d(x,O_n) \in Q\}$ is dense in $\overline{D_n}$ and all its elements are periodic, and hence do belong to the Mather set $\mathcal{M}_0$. Since $\mathcal{M}_0$ is closed, we deduce that $\overline{D_n} \subset \mathcal{M}_0$. This terminates the proof of the identity (8.2.31).

For the projected Mather set $\mathcal{M}$, it suffices to observe that on the determined $\mathcal{M}_0$, we have $f_t = \mathcal{R}_t$. Given that the Mather set is $f_t$-invariant, the result becomes clear. □

We note that if we set $p_n = \prod_{k=0}^{n} \rho_k$, then $\phi_L^{p_n}$ converges to the identity on $\mathcal{M}_0 setminus \bigcup_n D_n$. Moreover, a direct application of the properties of the Lemma 8.1.15 shows that the sets $D_n$ for all $x \in M$ and $y,z \in D_n$, then $\overline{h}(x,y) = \overline{h}(x,z)$ so that the sets $D_n$ are not detected by the Peierls barrier, and in particular by the generalized Peierls barrier $\underline{k}$ used in Theorem 7.1.2 to describe the non-wandering set $\Omega(\mathcal{T})$. Consequently, our framework will meet the assumptions of the uniformly $\underline{p}$-recurrent case studied in Section 7.4.3.

In order to state the corresponding result, we need to introduce a final set of definitions used in the representation formula of $\Omega(\mathcal{T})$. Recall from Section 7.3.1 the definition and the properties of the $\underline{p}$-Peierls Barrier $h^{\underline{p}} = \liminf_n h^{p_n}(x,y)$.

We introduce the map $d_{\underline{p}} : M \times M \to \mathbb{R}_{\geq 0}$ defined by

$$d_{\underline{p}}(x,y) = h^{\underline{p}}(x,y) + h^{\underline{p}}(y,x) \tag{8.2.33}$$

**Remark 8.2.23.** As seen for the classical Peierls barrier in Proposition 5.2.14, the $\underline{p}$-barrier $h^{\underline{p}}(x,\cdot)$ is a viscosity solution for all point $x$ in $M$. Moreover, for any $\underline{p}$-recurrent point $x$ of the Mather set $\mathcal{M}_0$ with a $\underline{p}$-recurrent lift $\tilde{x} \in \tilde{\mathcal{M}}_0$ under the Lagrangian flow $\phi_L$, then the map $h^{\underline{p}}(x,\cdot)$ is a $\underline{p}$-recurrent viscosity solution. Its recurrence speed is controlled by $dist\big(\tilde{x}, \phi_L^{p_n}(\tilde{x})\big)$ or equivalently by $dist(x, f_{p_n}(x))$. (See Proposition 7.3.5 for a proof of these non-trivial facts).

In our case, if $x$ belongs to $D_n$, then we can take $x'$ be in the boundary $\partial D_n$ of $D_n$. Hence, the point $x'$ is $\underline{p}$-recurrent under the projected Lagrangian flow $f_t$, and by Property 3 of Lemma 8.1.15, we have $h^{\underline{p}}(x,\cdot) = h^{\underline{p}}(x',\cdot)$ which is a $\underline{p}$-recurrent viscosity solution with recurrence speed controlled by $dist\big(x', f_{p_n}(x')\big)$.

Consequently, we deduce that in our case, for all $x \in \mathcal{M}_0$, the map $h^{\underline{p}}(x,\cdot)$ is a $\underline{p}$-recurrent viscosity solution with recurrence speed uniformly controlled by $dist\big(f_{p_n|(\mathcal{M}_0 \setminus \bigcup_n D_n)}, Id\big)$.

**Proposition 8.2.24.** *The map $d_{\underline{p}} : \mathcal{M}_0 \times \mathcal{M}_0 \to \mathbb{R}_{\geq 0}$ is a pseudometric on $\mathcal{M}_0$.*

*Proof.* The symmetry is clear.
*Reflexiveness.* Let $x$ be a point of $\mathcal{M}_0$. If $x$ belongs to $D_n$ for some integer $n$, then Property 3 of Lemma 8.1.15 asserts that $h^{\underline{p}}(x,x) = 0$. If not, then $x$ is $\underline{p}$-recurrent under the projected Lagrangian flow. We consider the $x(t)$ of the Mather set $\mathcal{M}$ starting at $x$. Then, Proposition 5.2.4 asserts that it is calibrated by any weak-KAM solution $u$. We get from the liminf property (7.3.5) of the Barrier that

$$h^{\underline{p}}(x,x) = \liminf_n h^{p_n}(x,x) = \liminf_n h^{p_n}(x, x(p_n)) = \lim_n u(x(p_n)) - u(x) = 0$$

*Triangular Inequality.* Using the first remark above and the triangular inequality (5.2.10) we get

$$\begin{aligned} h^{\underline{p}}(x,z) &= \lim_n h^{\underline{p}+p_n}(x,z) = \lim_n \liminf_k h^{p_k+p_n}(x,z) \\ &\leq \liminf_n \liminf_k h^{p_k}(x,y) + h^{p_n}(y,z) \\ &= \liminf_k h^{p_k}(x,y) + \liminf_n h^{p_n}(y,z) \end{aligned}$$

$$= h^{\underline{p}}(x,y) + h^{\underline{p}}(y,z)$$

□

This proposition allows to define the notion of $\underline{p}$-static classes as follows

**Definition 8.2.25.** 1. We set $\sim$ to be the equivalence relation in $\mathcal{M}_0$ given by

$$x \sim y \Longleftrightarrow d_{\underline{p}}(x,y) = 0 \tag{8.2.34}$$

2. The *$\underline{p}$-static classes* are the equivalence classes of the equivalence relation $\sim$. We denote by $\mathbb{M}_{\underline{p}}$ the set of $\underline{p}$-static classes represented by elements of $\mathcal{M}_0$ so that we have the inclusion $\mathbb{M}_{\underline{p}} \subset \mathcal{M}_0 \smallsetminus \bigcup_n D_n$.
3. A map $\psi : \mathbb{M}_{\underline{p}} \to \mathbb{R}$ is said *$\underline{p}$-dominated* if for all $x$ and $y$ in $\mathbb{M}_{\underline{p}}$, we have

$$\psi(y) - \psi(x) \leq h^{\underline{p}}(x,y) \tag{8.2.35}$$

We denote by $Dom_{\underline{p}}(\mathbb{M}_{\underline{p}})$ the set of dominated maps $\psi$ on $\mathbb{M}_{\underline{p}}$.

The generalized representation formula is stated as follows

**Theorem 8.2.26.** *If $\phi^{p_n}_{L|\mathbb{M}_{\underline{p}}}$ converges uniformly to the identity on $\mathbb{M}_{\underline{p}}$, then we have a bijection*

$$\begin{array}{ccc} Dom_{\underline{p}}(\mathbb{M}_{\underline{p}}) & \longrightarrow & \Omega(\mathcal{T}) \\ \psi & \longmapsto & \inf\limits_{y \in \mathbb{M}_{\underline{p}}} \{\psi(y) + h^{\underline{p}}(y,\cdot)\} \end{array} \tag{8.2.36}$$

*Moreover, every element $v$ of $\Omega(\mathcal{T})$ is $\underline{p}$-recurrent with uniform recurrence over $\Omega(\mathcal{T})$.*

We will simply apply this theorem to the constructed Mañé Lagrangian $L$ defined in (8.1.14). To do so, we first need to identify the set $\mathbb{M}_{\underline{p}}$ of $\underline{p}$-static classes and then, to better understand the $\underline{p}$-barrier $h^{\underline{p}}$. This is done by an immediate application of the properties of Proposition 8.1.12 and Lemma 8.1.15. Recall the notation of the different points introduced in (8.1.2).

**Proposition 8.2.27.** *We have*

*i. In dimension $d = 2$,*

$$\mathbb{M}_{\underline{p}} = \left( \bigcup_{\substack{n \geq 0 \\ 0 \leq i < \rho_n}} \{x^i_n\} \right) \cup \left( \bigcup_{n \geq 0} \{y_n, z^+_n\} \right) \cup \{z_\infty\} \tag{8.2.37}$$

*ii. In dimension $d \geq 3$,*

$$\mathbb{M}_{\underline{p}} = \left( \bigcup_{\substack{n \geq 0 \\ 0 \leq i < \rho_n}} \{x^i_n\} \right) \cup \left( \bigcup_{n \geq 0} \{y_n\} \right) \cup \{z_\infty\} \tag{8.2.38}$$

We define the set $\mathbb{M}'_{\underline{p}} = \mathbb{M}_{\underline{p}} \smallsetminus \{z_\infty\}$. Since $\mathbb{M}'_{\underline{p}}$ is dense in $\mathbb{M}_{\underline{p}}$, and due to the continuity of $\underline{p}$-dominated maps $\psi : \mathbb{M}_{\underline{p}} \to \mathbb{R}$, these maps are determined by their images in $\mathbb{M}'_{\underline{p}}$.

Notice that all points $x$ of $\mathcal{M}_0$ and $\mathbb{M}_{\underline{p}}$ are periodic, with integer periods, under the flow $f_t$. Consider the map $\rho : \mathcal{M}_0 \to \mathbb{N}_{\geq 1}$ which associates to each points, its (positive) integer period in $\mathbb{M}_{\underline{p}}$. More precisely,

$$\rho(x) = \rho_x = \begin{cases} \rho_n & \text{if } x = x_n^i,\ i = 0, .., \rho_n - 1 \\ 1 & \text{otherwise} \end{cases} \tag{8.2.39}$$

**Remark 8.2.28.** There is a subtlety about the points $y_n$. Note that all element $x$ of $\mathcal{M}_0 \smallsetminus \{y_n\}_{n\geq 0}$ are $\rho_x$-periodic under the flow $f_t$. However, this does not hold for the points $y_n$. Indeed, we have $\rho_{y_n} = 1$ and the point $y_n$ is not 1-periodic under the flow $f_t$. However, its $\underline{p}$-static class $\bar{y}_n$ is a fixed point of $f_{1|\mathbb{M}_{\underline{p}}}$ i.e. the points $y_n$ and $f_1(y_n)$ belong to the same $\underline{p}$-static class $\bar{y}_n$.

Furthermore, we can prove using Lemma 8.1.15 that

$$h^{\rho_n\infty + k}(y_n, \cdot) = h^{\rho_n\infty}(f_k(y_n), \cdot) = h^{\rho_n\infty}(y_n, \cdot)$$

which implies

$$h^{\rho_n\infty}(y_n, \cdot) = h^{\infty}(y_n, \cdot) = h^{\rho_{y_n}\infty}(y_n, \cdot)$$

This justifies the reason of taking $\rho(y_n) = 1$.

**Proposition 8.2.29.** *For every point $x$ in $\mathcal{M}_0$, the $\underline{p}$-barrier $h^{\underline{p}}(\cdot, x, \cdot)$ and the $\rho_x$-barrier $h^{\rho_x\infty}(\cdot, x, \cdot)$ do coincide on $\mathbb{R} \times M$.*

*Proof.* Fix an element $(t, x, y)$ in $\mathbb{R} \times \mathcal{M}_0 \times M$. Observe from Remark 8.2.28 that if $x = y_n$, then taking $\rho_{y_n} = 1$ or $\rho_n$ makes no difference. Thus, we can assume that $x$ is $\rho_x$-periodic under the flow $f_s$.

For $n \geq 0$ sufficiently large, the definition of $p_n$ allows us to consider an integer $p'_n$ such that $p_n = \rho_x.p'_n$. Then, we have

$$\begin{aligned} h^{\underline{p}+t}(x, y) &= \liminf_n h^{p_n + t}(x, y) = \liminf_n h^{\rho_x.p'_n + t}(x, y) \\ &\geq \liminf_n h^{\rho_x.n + t}(x, y) = h^{\rho_x\infty + t}(x, y) \end{aligned}$$

For the inverse inequality, fix an integer $m \geq 1$ and let $n$ be such that $p_n \geq \rho_x.m$. The Tonelli Theorem 5.1.9 guarantees the existence of a curve $\gamma_1 : [0, \rho_x.m + t] \to M$ from $x$ to $y$ such that $h^{\rho_x.m+t}(x, y) = A_L(\gamma_1)$. Now let $\gamma_2 : [0, p_n - \rho_x.m]$ be the curve defined by $\gamma_2(s) = f_s(x)$ so that it has null action $A_L(\gamma_2) = 0$. Since $x$ is $\rho_x$-periodic by the flow $f_t$, and the integer $\rho_x$ divides $p_n - \rho_x.m$, we deduce that $\gamma_2(0) = \gamma_2(p_n - \rho_x.m) = x$ and that $\gamma_2$ is a loop. Then, we can concatenate the two curves $\gamma_2$ and $\gamma_1$ to obtain a third one $\gamma : [0, p_n + t] \to M$ connecting $x$ to $y$. Hence, we get

$$h^{\rho_x.m+t}(x,y) = A_L(\gamma_1) = A_L(\gamma_1) + A_L(\gamma_2) = A_L(\gamma) \geq h^{p_n+t}(x,y)$$

Taking the liminf on $m$ and $k$, we derive the desired inequality

$$h^{\rho_x\infty+t}(x,y) \geq h^{\underline{p}+t}(x,y)$$

which concludes the proof. □

**Remark 8.2.30.** Using this proposition, we can express the constructed recurrent viscosity solutions $u$ defined in (8.2.1) as $u(x) = \inf_{n\geq 0}\{h^{\underline{p}}(x_n, x)\}$.

The properties drawn from Propositions 8.2.27 and 8.2.29 above finalize the application of Theorem 8.2.26 to the Mañé Lagrangian $L$, yielding

**Corollary 8.2.31.** *We have a bijective map*

$$\begin{array}{ccc} Dom_{\underline{p}}(\mathbb{M}_{\underline{p}}) & \longrightarrow & \Omega(\mathcal{T}) \\ \psi & \longmapsto & \inf\limits_{y\in\mathbb{M}'_{\underline{p}}}\{\psi(y) + h^{\rho_y\infty}(y,\cdot)\} \end{array} \tag{8.2.40}$$

*where the structure of* $\mathbb{M}'_{\underline{p}} = \mathbb{M}_{\underline{p}} \setminus \{z_\infty\}$ *is detailed in Proposition 8.2.27.*

We were able in the previous Subsection 8.2.2 to describe the dynamics of the Lax-Oleinik operator $\mathcal{T}$ on $\omega(u)$ for the initial date $u$ defined in (8.2.1). The following is a partial extension to the whole non-wandering set $\Omega(\mathcal{T})$.

**Proposition 8.2.32.** *For all recurrent viscosity solution $v$ in $\Omega(\mathcal{T})$, the Lax-Oleinik operator $\mathcal{T}$ restricted to the $\omega$-limit set $\omega(v)$ is a factor of an odometer map $\tau$.*

*Proof.* Let $v$ be an element of $\Omega(\mathcal{T})$ represented by the $\underline{p}$-dominated map $\psi : \mathbb{M}_{\underline{p}} \to \mathbb{R}$. Following the proof of Proposition 8.2.2, we get that for all non-negative integer $k \geq 0$,

$$\mathcal{T}^k v(x) = \inf_{y\in\mathbb{M}_{\underline{p}}}\{\psi(y) + h^{\rho_y\infty+k}(y,\cdot)\} \in \Omega(\mathcal{T}) \tag{8.2.41}$$

As in the proof of Theorem 8.2.15, we will characterize every element of $\mathrm{Orb}(\mathcal{T}, v)$ by the coefficients $k$ involved in the $(\rho_y, k)$-barriers $h^{\rho_y\infty+k}$ intervening in (8.2.41). For all $y$ in $\mathbb{M}_{\underline{p}}$ such that $\rho_y = 1$, we set $k_y = 0$. The other points $y \in \mathbb{M}_{\underline{p}}$ being in the $\{x_n^i\ ;\ 0 \leq i \leq \rho_n - 1,\ n \geq 0, \}$, we deduce that the remaining coefficients $k_y$ belong to the space $Z := \prod_{n\geq 0}(\mathbb{Z}/\rho_n\mathbb{Z})^{\rho_n}$. In other words, every element of the orbit $\mathrm{Orb}(\mathcal{T}, v)$ under $\mathcal{T}$ can be seen as an element of $Z$.

We endow the space $Z$ with a metric analogously to (8.2.17). And we consider the continuous map $\varphi$ defined by

$$\begin{array}{rrcl} \varphi : & Z = \prod_{n\geq 0}(\mathbb{Z}/\rho_n\mathbb{Z})^{\rho_n} & \longrightarrow & \Omega(\mathcal{T}) \\ & \underline{k} = (k_{x_n^i})_{i,n} & \longmapsto & \inf\limits_{y\in\mathbb{M}_{\underline{p}}} \{\psi(y) + h^{\rho_y\infty + k_y}(y,\cdot)\} \end{array} \tag{8.2.42}$$

Let $\tau$ be the odometer map on $Z$ given by

$$\begin{array}{rrcl} \tau : & Z & \longrightarrow & Z \\ & \underline{k} = (k_{x_n^i})_{i,n} & \longmapsto & \underline{k} + \underline{1} = (k_{x_n^i} + 1)_{i,n} \end{array} \tag{8.2.43}$$

Then, the identity (8.2.41) gives rise to the commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{\tau} & Z \\ \big\downarrow \varphi & & \big\downarrow \varphi \\ \Omega(\mathcal{T}) & \xrightarrow{\mathcal{T}} & \Omega(\mathcal{T}) \end{array} \tag{8.2.44}$$

Note that $\varphi(\underline{0}) = v$. Hence, by continuity of the maps involved in the diagram, their restrictions to the set $Z_0 = \overline{\{\tau^n(\underline{0})\ ;\ n \in \mathbb{N}\}}$ yields the new diagram

$$\begin{array}{ccc} Z_0 & \xrightarrow{\tau} & Z_0 \\ \big\downarrow \varphi & & \big\downarrow \varphi \\ \omega(v) & \xrightarrow{\mathcal{T}} & \omega(v) \end{array} \tag{8.2.45}$$

where the map $\varphi : Z_0 \to \omega(v)$ is onto. Therefore, $\mathcal{T}_{|\omega(v)}$ is a factor of an odometer map $\tau : Z_0 \to Z_0$. □

**Remark 8.2.33.**

1. The Theorem 8.0.2 is a direct consequence of the proof above.
2. This proof shows that the behaviour of $\mathcal{T}_{|\omega(v)}$ cannot be more complicated than that of $\tau_{|Z_0}$. Moreover, it is possible to obtain the entire dynamics of the odometer $\tau_{|Z_0}$ as seen in Theorem 8.2.15 and Corollary 8.2.21 where we got $Z_0 \simeq \omega(u)$ and $\tau_{|Z_0}$ is conjugate to $\mathcal{T}_{|\omega(u)}$.
3. When comparing the outcomes of the last two subsections, we can ask a natural question : Is it possible to determine the exact odometer acting on $\omega(v)$ ? This proves more intricate than in the case of $u$ due to the non-injectivity of the the map $\varphi$ constructed in the proof above. Even the fact that $\omega(v)$ is Cantor space isn't that clear.
4. However, for a generic $\underline{p}$-dominated map $\psi$ where the domination is everywhere strict, and if we impose a non symmetry condition between the values of $\{\psi(x_n^i)\}_i$, then it becomes possible to ensure the injectivity of the constructed map $\varphi$. Consequently,

the map $\varphi$ is a conjugacy between $\mathcal{T}_{|\omega(u)}$ and the odometer $\tau_{|Z_0}$ and we deduce that $\omega(v)$ is a Cantor set homeomorphic to $Z_0$.

## 8.3 Construction of a Smooth Non-Periodic Recurrent Viscosity Solution

In Section 8.2, we constructed a Lipschitz viscosity solution $u(t,x)$ of the Hamilton-Jacobi equation (5.1.8) that is recurrent and non-periodic. In this section, we undertake the proof of Theorem 8.0.1 and refine our choice of $u$ to achieve local $C^\infty$ regularity.

### 8.3.1 An Informal Preliminary Discussion.

The initial data $u$ chosen in (8.2.1) to obtain a recurrent, non-periodic viscosity solutions was

$$u(x) = \inf_{n\geq 0}\{h^{\rho_n\infty}(x_n, x)\} \tag{8.3.1}$$

where $x_n = x_n^0$ were defined in (8.1.2). However, there was no control over the sets of realization of the infimums and their boundaries where non-differentiability is very likely. Moreover, it has been established in Section 8.2.2 that all the recurrent viscosity solutions for the studied Mañé Lagrangian $L$ defined in (8.1.14) are of the form

$$u_c(x) = \inf_{y\in\mathbb{M}}\{c_y + h^{\rho_y\infty}(y,x)\}$$

where $\mathbb{M}$ is a well chosen subset of the Mather set $\mathcal{M}_0$ and $\rho_y$ are positive integers associated with the points $y$. Therefore, the potential regularization of $u$ should take this form.

For simplicity, let us fix an integer $n \geq 0$ and focus on a viscosity solution, simpler than (8.2.1), with initial data $v$ given by

$$v(x) = \inf\{h^\infty(x_n, x), h^\infty(z_n, x)\} \tag{8.3.2}$$

where $z_n$ is an arbitrary point of $\partial C_n$. Let us look at its possible regularity in $C_n$.

1. There is a significant risk of irregularity at the boundary between the two domains of the infimum, where $h^\infty(x_n, x) = h^\infty(z_n, x)$. However, this risk vanishes if the equality occurs on $\partial B_n$, which belongs to the Mather set where differentiability has been guaranteed by Remark 7.2.2. This concern can be dealt with by taking a modification of $v$ of the form

$$v_c(x) = \inf\{c_n + h^\infty(x_n, x), h^\infty(z_n, x)\} \tag{8.3.3}$$

where the constants $c_n$ are to be well chosen.

2. Let us look at the regularity of $h^\infty(x_n, x)$ in $B_n^0$. Suppose that $h^\infty(x_n, x)$ is "realized" or calibrates two distinct curves $\gamma_1, \gamma_2 : (-\infty, 0] \to M$ linking $x_n$ at $t \to -\infty$ to $x$ at $t = 0$.

   Being minimizing, these curves do follow the Lagrangian flow, implying $\dot\gamma_1(0) \neq \dot\gamma_1(0)$. However, if $h^\infty(x_n, x)$ is regular at $x$, the regularity Theorem 5.1.20 imposes that $d_x h^\infty(x_n, x) = \partial_v L(x, \dot\gamma_1(0))$ and $d_x h^\infty(x_n, x) = \partial_v L(x, \dot\gamma_2(0))$ which are distinct due to the convexity of $L$.

   As a result, a condition for regularity is to have one and only one minimizing curve that goes from $x_n$ to $x$, and this is the main reason why we imposed the various symmetries listed in Remarks 8.1.2 and 8.1.5 during the construction. The proofs will heavily rely on these.

Let's now delve into the proof. We begin by addressing the choice of the constants $c_n$.

### 8.3.2 Choice of the initial data $u$

As discussed in Subsection 8.3.1, we will proceed to a good choice of constants $c_n$, $n \geq 0$, ensuring that the viscosity solution $u_c$ defined by

$$u_c(x) = \inf_{n \geq 0} \{c_n + h^{\rho_n \infty}(x_n, x)\} \tag{8.3.4}$$

remains non-periodic, recurrent, and becomes regular. Additionally, we introduce the map $u_c^n$ defined by

$$u_c^n(x) = \inf_{k \neq n} \{c_k + h^{\rho_k \infty}(x_k, x)\} \tag{8.3.5}$$

**Proposition 8.3.1.** *There exist a sequence of real constants $c_n$ such that for the the associated solution $u_c$ we have*

*i. In the sets $B_n^0$, $u_c(x) = c_n + h^{\rho_n \infty}(x_n, x)$.*

*ii. In the sets $A_n$,*

- *(2D case) Either $u_c(x)$ is locally constant.*
- *(3D case and above) Or $u_c(x) = u_c^n(x) = h^\infty(z_\infty, x)$, where the point $z_\infty$ has been introduced in (8.1.2).*

*iii. In the sets $B_n \smallsetminus B_n^0$, $u_c(x)$ is constant.*

*iv. In the set $D$, $u_c(x)$ is locally constant.*

**Proposition 8.3.2.** *For the same real constants $c_n$ of Proposition 8.3.1, we have for all time $t \in \mathbb{R}$,*

*i. In the sets $\mathcal{R}_t(B_n^0)$, $u_c(t, x) = c_n + h^{\rho_n \infty + t}(x_n, x)$.*

*ii. In the sets $A_n$,*

— *(2D case) Either $u_c(t,x)$ is locally constant in $(t,x)$.*

— *(3D case and above) Or $u_c(t,x) = u_c^n(t,x) = h^{\infty+t}(z_\infty, x)$.*

*iii. In the sets $B_n \smallsetminus \mathcal{R}_t(B_n^0)$, $u_c(t,x)$ is constant in $(t,x)$.*

*iv. In the set $D$, $u_c(t,x)$ is locally constant.*

In these propositions, locally constant can be replaced by constant on the connected components of the considered sets.

Recall from Remark 8.1.1 that the 2D case and the higher-dimensional case present some topological differences. This adds a bit of intricacy to the selection of the constants $c_n$ which can be overtaken by separating the cases.

### Dimension $d \geq 3$ case

We prove Proposition 8.3.1 in the case of dimension $d \geq 3$. Recall from (8.1.2) that we fixed a point $z_\infty$ in $\overline{D}$ and points $y_n$ in $\partial B_n$. We set

$$c_n = h^\infty(z_\infty, y_n) - h^{\rho_n \infty}(x_n, y_n) \tag{8.3.6}$$

and we will show that these constants are convenient.

But first, note that the connectedness of the set $D$ in this higher-dimensional case leads to the following lemma, which slightly simplifies the current scenario.

**Lemma 8.3.3.** *For all $x \in D$, $u_c^n(x) = 0$ and for all $x \in C_n$, $u_c^n(x) = h^\infty(z_\infty, x)$.*

*Proof.* Let $x$ be a fixed point of $C_n$ and fix an integer $k \neq n$. Let us evaluate $h^{\rho_k \infty}(x_k, x)$. The set $\overline{D'}$ separates $x$ and $x_k$. Then, applying Property 4 of Lemma 8.1.15 applied to $F = \overline{D'}$ followed by an application of the Property 2 to the 1-periodic point $z_\infty$, yields

$$\begin{aligned} h^{\rho_k \infty}(x_k, x) &= h^{\rho_k \infty}(x_k, z_\infty) + h^{\rho_k \infty}(z_\infty, x) \\ &= h^{\rho_k \infty}(x_k, z_\infty) + h^\infty(z_\infty, x) \end{aligned} \tag{8.3.7}$$

Additionally, the set $\partial B_k$ separates the points $x_k$ and $z_\infty$ and the application of the same Lemma 8.1.15 gives

$$h^{\rho_k \infty}(x_k, z_\infty) = h^{\rho_k \infty}(x_k, y_k) + h^\infty(y_k, z_\infty)$$

Moreover, applying Property 3 of Lemma 8.1.15 to $F = \overline{D} \supset \partial C_n$, we get that $h^\infty(\cdot, \partial C_n) = h^\infty(\cdot, \overline{D})$ is well defined. Hence, the liminf property (7.3.5) applied to any curve $f_t(y)$,

$y \in A_k$, which $\alpha$ and $\omega$-limit sets respectively belong to $\partial B_k$ and $\partial C_k \subset \overline{D}$ yields

$$h^\infty(y_k, z_\infty) = h^\infty(\partial B_n, \partial C_n) \le \liminf_i A_L(f_t(y)_{|[-i,i]}) = 0 \tag{8.3.8}$$

This implies the equality $h^{\rho_k \infty}(x_k, z_\infty) = h^{\rho_k \infty}(x_k, y_k)$.
Gathering the identities leads to

$$\begin{aligned} u_c^n(x) &= \inf_{k \neq n} \{c_k + h^{\rho_k \infty}(x_k, x)\} \\ &= \inf_{k \neq n} \{c_k + h^{\rho_k \infty}(x_k, z_\infty) + h^\infty(z_\infty, x)\} \\ &= u_c^n(z_\infty) + h^\infty(z_\infty, x) \end{aligned}$$

with

$$\begin{aligned} u_c^n(D) = u_c^n(z_\infty) &= \inf_{k \neq n} \{c_k + h^{\rho_k \infty}(x_k, z_\infty)\} \\ &= \inf_{k \neq n} \{c_k + h^{\rho_k \infty}(x_k, y_k)\} \\ &= \inf_{k \neq n} \{h^{\rho_k \infty}(z_\infty, y_k)\} \end{aligned}$$

Moreover, the regularity of the barriers, as stated in Proposition 5.2.14, and another application of Lemma 8.1.15 provide

$$0 \le h^\infty(z_\infty, y_k) = h^\infty(z_\infty, y_k) - h^\infty(z_\infty, \partial C_k) \le \kappa_1.d(y_k, \partial C_k) = \kappa_1.\delta_k \to 0 \quad \text{as } k \to \infty$$

Thus, $u_c^n(D) = \inf_{k \neq n}\{h^\infty(z_\infty, y_k)\} = 0$ and we obtain the wanted result $u_c^n(x) = h^\infty(z_\infty, x)$. □

We can now conclude the proof of Propositions 8.3.1 8.3.2 for dimensions greater than 3.

**Proposition 8.3.4.** *The constants $c_n$ defined in (8.3.6) do answer the requirements of Proposition 8.3.1. More precisely,*

*i. For all $x \in B_n^0$, $c_n + h^{\rho_n \infty}(x_n, x) \le h^\infty(z_\infty, x)$.*

*ii. For all $x \in A_n$, $c_n + h^{\rho_n \infty}(x_n, x) \ge h^\infty(z_\infty, x)$.*

*iii. For all $x \in C_n \setminus (A_n \cup B_n^0)$, $c_n + h^{\rho_n \infty}(x_n, x) = h^\infty(z_\infty, x) = h^\infty(z_\infty, y_n)$.*

*iv. For all $x \in D$, $u_c(x) = 0$.*

*Proof.* *iv.* This is due to the Lemma 8.3.3 which states that for all $x \in D$, $u_c^n(x) = 0$ and $u_c(x) = \inf_{n \ge 0} u_c^n(x) = 0$.

*iii.* This third point is the most straightforward. In fact, the proof of Lemma 8.3.3 and more precisely of (8.3.8) shows that $h^{\rho_n \infty}(x_n, \cdot)$ is constant equal to $h^{\rho_n \infty}(x_n, y_n)$ on

$C_n \smallsetminus (B_n^0)$. Similarly, $h^\infty(z_\infty, \cdot)$ is constant equal to $h^\infty(z_\infty, y_n)$ on $\overline{B_n}$. Thus, recalling the definition (8.3.6) of $c_n$, we get the equality between the two maps $c_n + h^{\rho_n \infty}(x_n, \cdot)$ and $h^\infty(z_\infty, \cdot)$ on $C_n \smallsetminus (A_n \cup B_n^0) = \overline{B_n} \smallsetminus B_n^0$.

*i.* Now we prove the first point. Let $x$ be a point of $B_n^0$. Using the triangular inequality (5.2.28) for the Peierls barrier, we get

$$h^{\rho_n \infty}(x_n, x) \leq h^{\rho_n \infty}(x_n, y_n) + h^{\rho_n \infty}(y_n, x)$$

However a similar computation to (8.3.8) shows that $h^{\rho_n \infty}(y_n, x) = 0$. Moreover, we just saw from the third point that $h^\infty(z_\infty, x) = h^\infty(z_\infty, y_n)$. Hence, we deduce that

$$c_n + h^{\rho_n \infty}(x_n, x) \leq c_n + h^{\rho_n \infty}(x_n, y_n) = h^\infty(z_\infty, y_n) = h^\infty(z_\infty, x)$$

The second point on $A_n$ is proved analogously. □

**Proposition 8.3.5.** *The constants $c_n$ defined in (8.3.6) do answer the requirements of Proposition 8.3.2. More precisely, for all time $t \in \mathbb{R}$,*

*i. For all $x \in \mathcal{R}_t(B_n^0)$, $c_n + h^{\rho_n \infty + t}(x_n, x) \leq h^{\infty + t}(z_\infty, x)$.*

*ii. For all $x \in A_n$, $c_n + h^{\rho_n \infty + t}(x_n, x) \geq h^{\infty + t}(z_\infty, x)$*

*iii. For all $x \in C_n \smallsetminus (A_n \cup \mathcal{R}_t(B_n^0))$, $c_n + h^{\rho_n \infty + t}(x_n, x) = h^{\infty + t}(z_\infty, x) = h^{\infty + t}(z_\infty, y_n) = h^\infty(z_\infty, \partial B_n)$*

*iv. For all $x \in D$, $u_c(t) = 0$.*

*Proof.* We prove that

$$h^{\infty + t}(z_\infty, \cdot) = h^{t, \infty + t}(z_\infty, \cdot) \quad \text{and} \quad h^{\rho_n \infty + t}(x_n, \cdot) = h^{t, \rho_n \infty + t}(\mathcal{R}_t(x_n), \cdot)$$

Indeed, we use the triangular inequality (5.2.29) to obtain

$$h^{\rho_n \infty + t}(x_n, x) \leq h^t(x_n, f_t(x_n)) + h^{t, \rho_n \infty + t}(f_t(x_n), x) = h^{t, \rho_n \infty + t}(\mathcal{R}_t(x_n), x)$$

and

$$h^{t, \rho_n \infty + t}(\mathcal{R}_t(x_n), x) \leq h^{t, \rho_n \infty}(\mathcal{R}_t(x_n), x_n) + h^{\rho_n \infty}(x_n, x) = h^{\rho_n \infty}(x_n, x)$$

where we used the fact that $f_t(x_n) = \mathcal{R}_t(x_n)$ is $\rho_n$-periodic and of null action. Similarly, we prove that for all $\rho_n$-periodic point $x$ in $C_n$,

$$h^{\infty + t}(z_\infty, \mathcal{R}_t x) = h^\infty(z_\infty, x) \quad \text{and} \quad h^{\rho_n \infty + t}(x_n, \mathcal{R}_t x) = h^{\rho_n \infty}(x_n, x)$$

Hence, we deduce that

$$\begin{aligned}
c_n &= h^\infty(z_\infty, y_n) - h^{\rho_n\infty}(x_n, y_n)\\
&= h^\infty(z_\infty, \partial B_n) - h^{\rho_n\infty}(x_n, \partial B_n)\\
&= h^{\infty+t}(z_\infty, \mathcal{R}_t(\partial B_n)) - h^{\rho_n\infty+t}(x_n, \mathcal{R}_t(\partial B_n))\\
&= h^{\infty+t}(z_\infty, \partial B_n) - h^{\rho_n\infty+t}(x_n, \partial B_n)\\
&= h^{t,\infty+t}(z_\infty, \partial B_n) - h^{t,\rho_n\infty+t}(x_n, \partial B_n)
\end{aligned}$$

Note that the barrier $h^{t,\infty+t}$ is the Peierls barrier associated to the Mañé Lagrangian associated to the vector field $X_{t+\tau}$ with flow $f_{t,\tau}$, which restriction to $C_n$ can be represented by a rotation of Figure 8.1 by $\mathcal{R}_t$. Therefore, the proof of Proposition 8.3.5 is analogous to that of 8.3.4.

The fact that

$$h^\infty(z_\infty, y_n) = h^\infty(z_\infty, \partial B_n) = h^{\infty+t}(z_\infty, \partial B_n) = h^{t,\infty+t}(z_\infty, \partial B_n)$$

gives the constance in $t$ and $x$ in the statement of Proposition 8.3.2. □

**Dimension $d = 2$ case**

We now turn our attention to the dimension 2 case. In the forthcoming lemmas and results, we will present concise proofs, omitting redundant details that closely resemble those in the higher-dimensional case. However, we will highlight and elaborate on the distinctions pertinent to the 2D scenario.

As mentioned in Remark 8.1.1, the main difference between the 2D and the higher dimensional case is the disconnectedness of the sets $D$, $D_n$ and $A_n$. These invalidate the Lemma 8.3.3 as a curve that goes from $z_\infty$ to a point $x$ must cross all the sets $A_n$ and $B_n$ between them.

Recall from (8.1.1) that the sets $A_n$ are divided into two connected components $A_n^\pm$, with $A_n^+$ possessing the larger $r$-coordinate. We also introduced in (8.1.2) the points $z_n^\pm \in \partial A_n^\pm \cap \partial C_n$ and the points $y_n \in \partial B_n$.

**Lemma 8.3.6.** *We have the equalities*

$$h^\infty(z_n^+, y_n) = h^\infty(z_n^-, y_n) = h^\infty(\partial C_n, y_n) \tag{8.3.9}$$

This is due to the symmetries of $X_t$ in $A_n$ which were listed in Remark 8.1.5. More precisely, $X_t$ is invariant by rotation in the $\theta$-coordinate and is symmetric with respect to the circle $O_n = \{r = r_n\}$. These symmetries allow for a reduction to the one-dimensional case.

The proof of this lemma is postponed to Subsection 8.3.3, where explicit computations will be carried out using calibrated curves.

We set the constants $c_n$ to be

$$c_n = -h^{\rho_n\infty}(x_n, y_n) \tag{8.3.10}$$

**Proposition 8.3.7.** *These constants $c_n$ do answer the requirements of Proposition 8.3.1. More precisely,*

*i. For all $x \in C_0$, $u_c^n(x) = h^\infty(z_n^-, x)$.*

*ii. For all $n \geq 1$ and for all $x \in C_n$, $u_c^n(x) = \min\{h^\infty(z_n^\pm, x)\}$.*

*iii. For all $x \in C_n$, $c_n + h^{\rho_n\infty}(x_n, x) \leq u_c^n(x)$.*

*iv. For all $x \in C_n \smallsetminus B_n^0$, $h^{\rho_n\infty}(x_n, x) = h^{\rho_n\infty}(x_n, y_n)$.*

*Proof.* *iv.* The last statement is the most straightforward and is proved in the same fashion as the last point of Proposition 8.3.4.

*ii.* We now direct our attention to the second point and fix an integer $n \geq 1$. Let $x$ be a point within $C_n$, and consider an integer $0 \leq k < n$. Given that the curves linking $y_k$ to $y_{k+1}$ must intersect $\partial A_k^-$ and $\partial A_{k+1}^+$, the application of Property 4 from Lemma 8.1.15 yields :

$$h^\infty(y_k, y_{k+1}) = h^\infty(y_k, z_k^-) + h^\infty(z_k^-, z_{k+1}^+) + h^\infty(z_{k+1}^+, y_{k+1}) = h^\infty(z_{k+1}^+, y_{k+1})$$

where all the null terms are deduced from the direction of the flow $f_t$ from one component $\partial A_i^\pm$ to another or from Property 3 of Lemma 8.1.15. Hence, since every curve linking $y_k$ to $z_n^+$ must the sets $\{r = r_i + \delta_i\} \subset \partial A_i^+$ for all $i = k+1, .., n-1$, we obtain

$$\begin{aligned} h^\infty(y_k, z_n^+) &= \sum_{j=k+1}^{n-1} h^\infty(y_{j-1}, y_j) + h^\infty(y_{n-1}, z_n^+) \\ &= \sum_{j=k+1}^{n-1} h^\infty(z_j^+, y_j) + 0 \end{aligned}$$

Again, since every curve linking $x_k$ to $z_n^+$ must intersect $\{r = r_k + \delta_k\} \ni y_k$ and $\{r = r_n - 2\delta_n\} \ni z_n$, we get

$$\begin{aligned} c_k + h^{\rho_k\infty}(x_k, x) &= c_k + h^{\rho_k\infty}(x_k, y_k) + h^\infty(y_k, z_n^+) + h^\infty(z_n^+, x) \\ &= \sum_{j=k+1}^{n-1} h^\infty(z_j^+, y_j) + h^\infty(z_n^+, x) \end{aligned} \tag{8.3.11}$$

It follows that

$$u_c^{<n}(x) := \inf_{k<n}\{c_k + h^{\rho_k\infty}(x_k, x)\} = c_{n-1} + h^{\rho_{n-1}\infty}(x_{n-1}, x) = h^\infty(z_n^+, x)$$

Similarly, we show that

$$u_c^{>n}(x) := \inf_{k>n}\{c_k + h^{\rho_k\infty}(x_k, x)\} = c_{n+1} + h^{\rho_{n+1}\infty}(x_{n+1}, x) = h^\infty(z_n^-, x)$$

The result follows from the identity $u_c^n = \min\{u_c^{<n}, u_c^{>n}(x)\}$.

*i.* In the case of $C_0$, we have $u_c^0 = u_c^{>n}(x)$.

*iii.* For this last inequality, one needs to notice that for all $x \in C_n$,

$$c_n + h^{\rho_n\infty}(x_n, x) \le c_n + h^{\rho_n\infty}(x_n, \partial C_n) = c_n + h^{\rho_n\infty}(x_n, y_n) = 0 \le u_c^n(x)$$

□

Working with the Mañé Lagrangian associated to $X_{t+\tau}$, we obtain

**Proposition 8.3.8.** *These constants $c_n$ do answer the requirements of Proposition 8.3.1. More precisely,*

*i. For all $x \in C_0$, $u_c^n(t,x) = h^{\infty+t}(z_n^-, x)$.*

*ii. For all $n \ge 1$ and for all $x \in C_n$, $u_c^n(x) = \min\{h^{\infty+t}(z_n^\pm, x)\}$.*

*iii. For all $x \in C_n$, $c_n + h^{\rho_n\infty+t}(x_n, x) \le u_c^n(t,x)$.*

*iv. For all $x \in C_n \smallsetminus B_n^0$, $h^{\rho_n\infty+t}(x_n, x) = h^{\rho_n\infty+t}(x_n, y_n)$.*

### Recurrence and Non-Periodicity of $u_c$

We verify that the chosen initial data corresponds to a recurrent, non-periodic viscosity solution $u_c(t,x)$. We saw in Section 8.2.2 that the form (8.3.5) of the chosen initial data $u_c$ corresponds to a recurrent viscosity solution of the Hamilton-Jacobi equation. Hence, we only need to confirm the non-periodicity.

**Non-Periodicity.** We claim that

(i) For $k = 0, \rho_n$, $\mathcal{T}^k u_c(x_n) = c_n$.

(ii) For $k = 1, ..., \rho_n - 1$, $\mathcal{T}^k u_c(x_n) > c_n$.

In fact, Proposition 8.2.2 and Lemma 8.2.7 yield

$$\mathcal{T}^k u_c(x) = \inf_{n\ge 0}\{c_n + h^{\rho_n\infty+k}(x_n, x)\} = \inf_{n\ge 0}\{c_n + h^{\rho_n\infty}(x_n^k, x)\} \tag{8.3.12}$$

A symmetric version of Propositions 8.3.4 and 8.3.7 featuring $x_n^i$ in $B_n^i$ instead of $x_n$ in $B_n^0$ results in :

(i) For $k = 0, \rho_n$,

$$\mathcal{T}^k u_c(x_n) = c_n + h^{\rho_n\infty}(x_n^k, x_n) = c_n + h^{\rho_n\infty}(x_n, x_n) = c_n$$

(ii) For all $k = 1, ..., \rho_n - 1$,

$$\mathcal{T}^k u_c(x_n) = c_n + h^{\rho_n \infty}(x_n^k, x_n) = c_n + h^{\rho_n \infty}(x_n^k, y_n) > c_n$$

Therefore, $(\mathcal{T}^k u_c)_{k \geq 0}$ cannot be periodic.

### 8.3.3 Proof of the $C^\infty$ Regularity of $u_c$

We begin the proof of Theorem 8.0.1 by establishing the $C^\infty$ regularity of the constructed recurrent, non-periodic viscosity solution $u_c$. This approach heavily relies on the differentiability along calibrated curves, as outlined in Theorem 5.1.20. Since the differential has a simple expression on these curves, higher regularity can be achieved through a well-behaved foliation of $M$ by calibrated curves. Thus, identifying these curves becomes crucial. The preliminary lemmas presented in the next subsection serve this purpose.

#### Preliminary Lemmas : Identifying Suitable Calibrated Curves

We will take advantage of the autonomy of the flow $g_t$ used to define $f_t$. Let us introduce the autonomous Mañé Lagrangian $L_Z : TM \to \mathbb{R}$ associated to the vector field $Z$ defined by

$$L_Z(x, v) = \frac{1}{2}\|v - Z(x)\|^2 \tag{8.3.13}$$

And we denote by $h_Z$ and $m_Z$ its relative barriers and Mañé potentials.

Analogously to Propisition 8.2.22, we get

**Proposition 8.3.9.** *The Mather set $\tilde{\mathcal{M}}_Z$ and its projection $\mathcal{M}_L$ to $M$ associated to $L_Z$ are $\tilde{\mathcal{M}}_Z = \mathbb{T}^1 \times \tilde{\mathcal{M}}_0$ and $\mathcal{M}_Z = \mathbb{T}^1 \times \mathcal{M}_0$*

**Lemma 8.3.10.** *A curve $\gamma : \mathbb{R} \to C_n$ is calibrated by $h^{\rho_n \infty}(x_n, \cdot)$ (resp. $h^\infty(z_\infty, \cdot)$) if and only if the curve $\sigma(t) = \mathcal{R}_t^{-1} \circ \gamma(t)$ is calibrated by $h_Z^{\rho_n \infty}(x_n, \cdot)$ (resp. $h_Z^\infty(z_\infty, \cdot)$).*

*Proof.* We prove the case of $h^{\rho_n \infty}(x_n, \cdot)$. We will show that for all times $t \in \mathbb{R}$ and all points $x \in C_n$,

$$h^{\rho_n \infty + t}(x_n, x) = h_Z^{\rho_n \infty + t}(x_n, \mathcal{R}_t^{-1} x) \tag{8.3.14}$$

We establish double inequality. Let $n_k$ be an increasing sequence of integers and $\sigma_k : [0, \rho_n n_k + t] \to M$ be a sequence of curves linking $x_n$ to $\mathcal{R}_t^{-1} x$ and such that $h_Z^{\rho_n \infty}(x_n, \mathcal{R}_t^{-1} x) = \lim_k A_{L_Z}(\sigma_k)$.

We aim to replace $\sigma_k$ by curves $\tilde{\sigma}_k$ with image in the closure $\overline{C}_n$ of $C_n$. We treat the case of dimension 3 or above. We consider the real numbers $s_k^\pm$ defined by

$$s_k^- := \inf\{\tau \in [0, \rho_n n_k + t] \mid \sigma_k(\tau) \notin C_n\} \quad \text{and} \quad s_k^+ := \sup\{\tau \in [0, \rho_n n_k + t] \mid \sigma_k(\tau) \notin C_n\}$$

Fix a smooth map $\zeta_k : [0,1] \to \partial C_n$ linking $\sigma_k(s_k^-)$ to $\sigma_k(s_k^+)$. Such map exists since in dimension 3 or above, the set $\partial C_n$ is connected. Then, for all integer $l \geq 1$, we set the curve $\sigma_{k,l} : [0, \rho_n(n_k + l) + t] \to \overline{C}_n$ to be given by

$$\sigma_{k,l}(\tau) = \begin{cases} \sigma_k(\tau) & \text{if } \tau \in [0, s_k^-] \\ \zeta_k\left(\frac{\tau - s_k^-}{s_k^+ - s_k^- + \rho_n l}\right) & \text{if } \tau \in [s_k^-, s_k^+ + \rho_n l] \\ \sigma_k(\tau - \rho_n l) & \text{if } \tau \in [s_k^+ + \rho_n l, \rho_n(n_k + l) + t] \end{cases} \tag{8.3.15}$$

Hence, we have

$$\begin{aligned} A_{L_Z}(\sigma_{k,l}) &= \int_0^{\rho_n(n_k+l)+t} \frac{1}{2}\|\dot{\sigma}_{k,l} - Z(\sigma_{k,l})\|^2 d\tau \\ &= \int_{[0,s_k^-]\cup[s_k^+ \rho_n n_k + t]} \frac{1}{2}\|\dot{\sigma}_k - Z(\sigma_k)\|^2 \, d\tau + \int_{s_k^-}^{s_k^+ + \rho_n l} \frac{1}{2}\|\dot{\sigma}_{k,l} - Z(\sigma_{k,l})\|^2 d\tau \\ &= A_{L_Z}(\sigma_{k|[0,s_k^-]\cup[s_k^+ \rho_n n_k]}) + \int_{s_k^-}^{s_k^+ + \rho_n l} \frac{1}{2(s_k^+ - s_k^- + \rho_n l)^2} \left\| \dot{\zeta}_k \left( \frac{\tau - s_k^-}{s_k^+ - s_k^- + \rho_n l} \right) \right\|^2 d\tau \\ &\leq A_{L_Z}(\sigma_k) + \frac{\|\dot{\zeta}_k\|^2}{2(s_k^+ - s_k^- + \rho_n l)} \longrightarrow 0 \quad \text{as } l \to +\infty \end{aligned}$$

where we used in the third equality that $Z$ is null in $\partial C_n$. Hence, by extracting simultaneously two increasing subsequences $n_{k_i}$ and $l_i$ of $n_k$ and $l$ such that $A_{L_Z}(\sigma_{k_i,l_i})$ converges, we obtain

$$\lim_i A_{L_Z}(\sigma_{k_i,l_i}) \leq \lim_i A_{L_Z}(\sigma_{k_i})$$

Thus, using the liminf property (7.3.5) of the Peierls barrier yields

$$h_Z^{\rho_n\infty+t}(x_n, \mathcal{R}_t^{-1}x) \leq \lim_i A_{L_Z}(\sigma_{k_i,l_i}) \leq \lim_i A_{L_Z}(\sigma_{k_i}) = h_Z^{\rho_n\infty+t}(x_n, \mathcal{R}_t^{-1}x)$$

and we deduce the equality $h_Z^{\rho_n\infty+t}(x_n, \mathcal{R}_t^{-1}x) = \lim_i A_{L_Z}(\sigma_{k_i,l_i})$ for curves $\sigma_{k_i,l_i}$ with images in $C_n$.

We get the same conclusion for the 2D case by setting $s_{k,j}^\pm$ being successive times of entering and exiting the connected components of $M \setminus C_n$.

We can now show the identity (8.3.14). For simplicity, we change the notation $\sigma_i$ for $\sigma_{k_i,l_i}$. We set for all $i$ the curve $\gamma_i(\tau) = \mathcal{R}_\tau \sigma_i(\tau)$. The evaluation of its velocity gives

$$\dot{\gamma}_i(\tau) = \frac{d}{dt}(\mathcal{R}_\tau \sigma_i)(\tau) = \frac{d\mathcal{R}_\tau}{d\tau}\sigma_i(\tau) + d\mathcal{R}_\tau.\dot{\sigma}_i(\tau) = Y_\tau(\gamma_i(\tau)) + d\mathcal{R}_\tau.\dot{\sigma}_i(\tau)$$

and using the form of $X_\tau$ given by (8.1.13), we get

$$\|\dot{\gamma}_i(\tau) - X_\tau(\gamma_i(\tau))\| = \|d\mathcal{R}_\tau.\dot{\sigma}_i(\tau) - d\mathcal{R}_\tau Z(\sigma_u(\tau))\| = \|\dot{\sigma}_i(\tau) - Z(\sigma_i(\tau))\|$$

and consequently, $A_L(\gamma_i) = A_{L_Z}(\sigma_i)$. Moreover, we have $\gamma_i(0) = \sigma_i(0) = x_n$ and by $\rho_n$-periodicity of $\mathcal{R}_\tau$ on $C_n$,

$$\gamma_i(\rho_n(n_{k_i} + l_i) + t) = \mathcal{R}_{\rho_n(n_{k_i}+l_i)+t}\sigma_i(\rho_n(n_{k_i} + l_i) + t) = \mathcal{R}_t\mathcal{R}_t^{-1}x = x$$

Hence, we the liminf Property (7.3.5) leads to

$$h^{\rho_n\infty+t}(x_n, x) \leq \liminf_i A_L(\gamma_i) = \lim_i A_{L_Z}(\sigma_i) = h_Z^{\rho_n\infty+t}(x_n, \mathcal{R}_t^{-1}x)$$

The inverse inequality is obtained analogously by showing that we can find $\gamma_i$ with image on $C_n$ such that $h^{\rho_n\infty+t}(x_n, x) = \lim_k A_L(\gamma_i)$. This ends the proof the identity (8.3.14). Application to calibrated curves $\gamma$ and $\sigma(t) = \mathcal{R}_t^{-1} \circ \gamma(t)$ yields

$$h_Z^{\rho_n\infty+t}(x_n, \sigma(t)) = h^{\rho_n\infty+t}(x_n, \gamma(t)), \quad h^{\infty+t}(z_\infty, \gamma(t)) = h_Z^{\infty+t}(z_\infty, \sigma(t)) \quad \text{and} \quad A_{L_Z}(\sigma) = A_L(\gamma)$$

These imply the result. □

The following Lemmas focus on identifying the calibrated curves of $h_Z^{\rho_n\infty}(x_n, \cdot)$ and $h_Z^\infty(\partial C_n, \cdot)$ respectively on $B_n^0$ and $A_n$.

**Lemma 8.3.11.** *Let $u$ be a periodic viscosity solution of the Hamilton-Jacobi equation (5.1.8) associated to $L_Z$ and consider for all point $(t, x)$ in $\mathbb{R} \times M$ a $u$-calibrated curve $\sigma_{t,x} : (-\infty, t] \to M$ with $\sigma_{t,x}(t) = x$. Then*

*i. If $x$ belongs $B_n^i$ then for all time $s \leq t$, $\sigma_{t,x}(s) \in B_n^i$.*

*ii. If $x$ belongs $A_n$ then for all time $s \leq t$, $\sigma_{t,x}(s) \in A_n$.*

Note that periodic viscosity solutions for $L_Z$ are stationary weak-KAM solutions due to Fathi's Convergence Theorem 7.1.3.

*Proof.* We only prove the case of $B_n^0$. The case of $A_n$ is done similarly. Let $x$ be a point of $B_n^0$. Arguing by contradiction, suppose that $\sigma_{t,x}$ exits the ball $B_n^0$. Then there must exist a time $s < t$ such that $\sigma_{t,x}(s) \in \partial B_n^0$. Since the curve $\sigma_{t,x}$ is calibrated by $u$, we deduce from Theorem 5.1.20 that $u$ is differentiable at $(s, \sigma_{t,x}(s))$ and $\partial_x u(s, \sigma_{t,x}(s)) = \partial_v L(s, \sigma_{t,x}(s), \dot\sigma_{t,x}(s))$.

Now recall from Proposition 8.3.9 that $\partial B_n^0 \subset \mathcal{M}_Z$. Thus, we get the inclusion $(s, \sigma_{t,x}(s)) \in \mathcal{M}_Z$ and there exists an element $v$ of $T_{\sigma_{t,x}(s)}M$ such that $(s, \sigma_{t,x}(s), v) \in \tilde{\mathcal{M}}_Z$. The Proposition 7.2.1, added to the regularity Theorem 5.1.20 on calibrated curves, shows that $u$ must be differentiable at $(s, \sigma_{t,x}(s), v)$ and $\partial_x u(s, \sigma_{t,x}(s)) = \partial_v L_Z(\sigma_{t,x}(s), v)$. We get $\partial_v L_Z(\sigma_{t,x}(s), \dot\sigma_{t,x}(s)) = \partial_v L_Z(\sigma_{t,x}(s), v)$. However, the Legendre map $\mathcal{L} : (\tau, y, w) \to (\tau, y, \partial_v L_Z(y, w))$ is a bijective in the Tonelli case (see [Fat08]). Hence, we get $v = \dot\sigma_{t,x}(s)$ and by the $\phi_{L_Z}$-invariance of the Mather set, we also get that $(\tau, \sigma_{t,x}(\tau)) = \big(\tau, \pi \circ \phi_{L_Z}^{s,\tau}(\sigma_{t,x}(s), v)\big)$ belongs to $\tilde{\mathcal{M}}_Z$ for all times $\tau \leq t$. The fact that $\sigma_{t,x}(s) \in \partial B_n^0$ implies that

$\sigma_{t,x}$ must belong in the connected component of $\mathcal{M}_Z$ that contains $\partial B_n^0$. This contradicts the fact that $x$ belongs to $B_n^0$. □

**Lemma 8.3.12.** *For all integer $n \geq 0$,*

1. *On $B_n^0$, consider the spherical coordinates $(\delta_B, \varphi_2^B, .., \varphi_d^B) = (\delta_B, \bar{\delta}_B)$.*
   *If $x \in B_n^0$ and $\sigma_x : (-\infty, t] \to M$ with $\sigma_x(t) = x$ is a curve calibrated by $h_Z^{\rho_n\infty}(x_n, \cdot)$, then $\sigma_x(t)$ has constant $\bar{\delta}_B$-coordinate, $\alpha_1(\sigma_x) = \{x_n\}$ and for all $s \leq t$*
   $$h_Z^{\rho_n\infty}(x_n, \sigma_x(s)) = h_Z^{\rho_n\infty+s}(x_n, \sigma_x(s)) = A_L(\sigma_{x|(-\infty,s]}) = \int_{-\infty}^{s} L(\tau, \sigma_x(\tau), \dot{\sigma}_x(\tau))\, d\tau \tag{8.3.16}$$
2. *On $A_n$, we have $A_n \subset C_n = O_n \times B^{d-1}$ where is a $d-1$-dimensional ball centered on $x_n$ and of radius $2\delta$. Using the spherical coordinates on $B^{d-1}$, we define the coordinates $(\theta, \delta_A, \varphi_2^A, .., \varphi_d^A) = (\delta_A, \bar{\delta}_A)$ on $A_n$.*
   *If $x \in A_n$ and $\sigma_x : (-\infty, t] \to M$ with $\sigma_x(t) = x$ is a curve calibrated by $h_Z^\infty(z_\infty, \cdot)$, then $\sigma_x(t)$ has constant $\bar{\delta}_A$-coordinate, $\alpha_1(\sigma_x) \subset \partial C_n$ and for all $s \leq t$*
   $$h_Z^{\infty}(z_\infty, \sigma_x(s)) = h_Z^{\infty+s}(z_\infty, \sigma_x(s)) = A_L(\sigma_{x|(-\infty,s]}) = \int_{-\infty}^{s} L(\tau, \sigma_x(\tau), \dot{\sigma}_x(\tau))\, d\tau \tag{8.3.17}$$

*Proof.* We only prove the first point, the second being analogous. Note that due to Fathi's Theorem 7.1.3, we have $h_Z^{\rho_n\infty+s} = h_Z^{\rho_n\infty} = h_Z^\infty$ is independent on time. Hence, we deduce from Property 3 of Proposition 5.2.14 that $h_Z^{\rho_n\infty}(x_n, \cdot)$ is a weak-KAM solution associated to $L_Z$. And since $x$ belongs to $B_n^0$, we deduce from Lemma 8.3.11 that $\sigma_x(\tau) \in B_n^0$ for all time $\tau \leq t$.

We first prove that $\alpha_1(\sigma_x) = \{x_n\}$. We saw that $\sigma_x$ is calibrated by a weak-KAM solution. Hence, Proposition 5.2.20 asserts that $\sigma_x$ is semi-static, and according to Proposition 5.2.25, its $\alpha$-limit $\alpha_1(\sigma_x)$ belongs to a single static class of $\mathbb{M}_Z$. Knowing that $\sigma_x$ has its image in the ball $B_n^0$, we deduce that $\alpha_1(\sigma_x) = \bar{x}_n = \{x_n\}$, or $\alpha_1(\sigma_x) \subset \partial B_n^0$.

Assume that $\alpha_1(\sigma_x) \subset \partial B_n^0$. There exists an increasing sequence of integers $k_i$ and a point $y \in \partial B_n^0$ such that $\sigma_x(-\rho_n k_i)$ converges to $y$. Taking the limit for $t = -\rho_n k_i$ in the calibration equation and using the positivity of the Lagrangian $L_Z$, we get

$$h_Z^{\rho_n\infty}(x_n, x) = h_Z^{\rho_n\infty}(x_n, y) + A_{L_Z}(\sigma_{x|(-\infty,0]}) > h_Z^{\rho_n\infty}(x_n, y) \tag{8.3.18}$$

the strict inequality is due to the strict positivity of $A_L(\sigma_{x|(-\infty,0]}) > 0$ since the flow $g_t$ of $Z$ is directed from $\partial B_n^0$ towards its center $x_n$, at the opposite of $\sigma_x$. However, an application of the triangular inequality (5.2.28) of the Peierls barrier gives

$$h_Z^{\rho_n\infty}(x_n, x) \leq h_Z^{\rho_n\infty}(x_n, y) + h_Z^{\rho_n\infty}(y, x)$$

We claim that $h_Z^{\rho_n\infty}(y,x) = 0$. Indeed, if $y'$ is a limit point of the sequence $g^{-1}_{\rho_n k}(x)$, we have $y' \in \partial B_n^0$ and

$$0 \le h_Z^{\rho_n\infty}(y,x) \le h_Z^{\rho_n\infty}(y,y') + h_Z^{\rho_n\infty}(y',x) \le h_Z^{\rho_n\infty}(y,y') + \liminf_k h_Z^{\rho_n k}(f^{-1}_{\rho_n k}(x),x) = 0+0$$

where we used Lemma 8.1.15 for the nullity of the first term and the fact that the flow $g_t$ has null action for the nullity of the second term. We finally get $h_Z^{\rho_n\infty}(x_n,x) \le h_Z^{\rho_n\infty}(x_n,y)$ which contradicts (8.3.18). Therefore, we deduce that $\alpha_1(\sigma_x) = \bar{x}_n = \{x_n\}$.

Let $k_i$ be an increasing integer sequence such that $\sigma_x(-\rho_n k_i)$ converges to $x_n$. Taking the limit in the calibration equation, we get

$$h_Z^{\rho_n\infty}(x_n,x) = h_Z^{\rho_n\infty}(x_n,x_n) + \lim_i A_L(\sigma_{x|(-\rho_n k_i,0]}) = A_L(\sigma_{x|(-\infty,0]})$$

Let us show that $\sigma_x$ has constant $\bar{\delta}_B$-coordinate. We adopt the notation $\sigma_x = (\sigma_\delta, \sigma_{\bar{\delta}})$ in the spherical coordinates. Let $\bar{\delta}_t$ be the $\bar{\delta}$-coordinate of $x = (x_\delta, x_{\bar{\delta}})$ in the spherical coordinates, and consider the curve $\tilde{\sigma}_x$ given by $\tilde{\sigma}_x = (\sigma_\delta, \bar{\delta}_t) \in \{\bar{\delta} = \bar{\delta}_t\}$. Recall from Remark 8.1.5 that, on $B_n^0$, the vector field $Z$ has null $\bar{\delta}$ coordinate and that its $\delta$-coordinate $Z_\delta(y)$ depends only on the $\delta$-coordinate of $y$, i.e $Z_\delta(\sigma_x(t)) = Z_\delta(\tilde{\sigma}_x(t))$. Hence, using the notation $\dot{\sigma}_x = \dot{\sigma}_\delta \overset{\perp}{+} \dot{\sigma}_{\bar{\delta}}$, we have for all $\tau \le t$,

$$\begin{aligned} L_Z(\sigma_x(\tau), \dot{\sigma}_x(\tau)) &= \frac{1}{2}\|\dot{\sigma}_x(\tau) - Z(\sigma_x(\tau))\|^2 = \frac{1}{2}|\dot{\sigma}_\delta(\tau) - Z_\delta(\sigma_x(\tau))|^2 + \frac{1}{2}\|\dot{\sigma}_{\bar{\delta}}(\tau)\|^2 \\ &= \frac{1}{2}\|\dot{\tilde{\sigma}}_x(\tau) - Z(\tilde{\sigma}_x(\tau))\|^2 + \frac{1}{2}\|\dot{\sigma}_{\bar{\delta}}(\tau)\|^2 = L_Z(\tilde{\sigma}_x(\tau), \dot{\tilde{\sigma}}_x(\tau)) + \frac{1}{2}\|\dot{\sigma}_{\bar{\delta}}(\tau)\|^2 \end{aligned}$$

We have $\alpha_1(\tilde{\sigma}_x) = \{x_n\}$. Thus, noting that $x = \sigma_x(t) = \tilde{\sigma}_x(t)$, we get

$$\begin{aligned} h_Z^{\rho_n\infty+t}(x_n, \sigma_x(t)) &= h_Z^{\rho_n\infty+t}(x_n, \tilde{\sigma}_x(t)) \\ &\le \int_{-\infty}^t L_Z(\tilde{\sigma}_x(\tau), \dot{\tilde{\sigma}}_x(\tau))\, d\tau \\ &\le \int_{-\infty}^t L_Z(\tilde{\sigma}(\tau), \dot{\tilde{\sigma}}(\tau)) + \frac{1}{2}\|\dot{\sigma}_{\bar{\delta}}(\tau)\|^2 d\tau \\ &= \int_{-\infty}^t L_Z(\sigma(\tau), \dot{\sigma}(\tau))\, d\tau = h_Z^{\rho_n\infty+t}(x_n, \sigma(t)) \end{aligned}$$

Therefore, there is equality everywhere and $\int_{-\infty}^t \frac{1}{2}\|\dot{\sigma}_{\bar{\delta}}(\tau)\|^2 d\tau = 0$. By continuity of $\dot{\sigma}_{\bar{\delta}}$, we deduce that it is null and that $\sigma_{\bar{\delta}}$ is constant. □

**Remark 8.3.13.** By Property 4 of Lemma 8.1.15, note that for all curve $\gamma$ in $C_n$, being calibrated by $h_Z^\infty(z_\infty,\cdot)$ is equivalent to being calibrated by $h_Z^\infty(\partial C_n,\cdot) = h_Z^\infty(z_\infty,\cdot)$ in dimension higher than 2 and by $h_Z^\infty(z_n^-,\cdot) = -h_Z^\infty(z_\infty,z_n^-) + h_Z^\infty(z_\infty,\cdot)$ in the 2D case.

Moreover, the result remains valid for the calibration by $h_Z^\infty(z_n^+,\cdot)$.

**Lemma 8.3.14.** *We denote by $\phi_{-Z}$ the flow associated to the vector field $-Z$.*

1. *For all $x \in B_n^0$, the curve $\sigma_x : \mathbb{R} \to B_n^0$ defined by $\sigma_x(t) = \phi_{-Z}^t(x)$ is calibrated by $h_Z^{\rho_n\infty}(x_n,\cdot)$.*
2. *For all $x \in A_n$, the curve $\sigma_x : \mathbb{R} \to A_n$ defined by $\sigma_x(t) = \phi_{-Z}^t(x)$ is calibrated by $h_Z^\infty(\partial C_n,\cdot)$.*

*Proof.* We prove the first point, the second being analogous. Let $t > 0$ be a positive time, and let $y = g_t(x) = \phi_{-Z}^t(x)$ be a point of $B_n^0$. Since $h^{\rho_n\infty}(x_n,\cdot)$ is a viscosity solution, we get from Proposition 5.1.17 the existence of a calibrated curve $\sigma : (-\infty, t] \to M$. Hence, we infer from Lemma 8.3.12 that $\alpha_1(\sigma) = \{x_n\}$, $\sigma$ has constant $\bar{\delta}$-coordinate, and that for all $s < t$

$$h_Z^{\rho_n\infty}(x_n,\sigma(s)) = \int_{-\infty}^s \frac{1}{2}\|\dot\sigma(\tau) - Z(\sigma(\tau))\|^2\, d\tau = \int_{-\infty}^s \frac{1}{2}|\dot\sigma_\delta(\tau) - Z_\delta(\sigma(\tau))|^2\, d\tau$$

Then,

$$\frac{d}{ds}h_Z^{\rho_n\infty}(x_n,\sigma(s)) = \frac{1}{2}\|\dot\sigma(s) - Z(\sigma(s))\|^2$$

Moreover, Theorem 5.1.20 shows that $h_Z^{\rho_n\infty}(x_n,\cdot)$ is differentiable at $\sigma(s)$ for all $s < t$, yielding

$$\begin{aligned}\frac{d}{ds}h_Z^{\rho_n\infty}(x_n,\sigma(s)) &= dh_Z^{\rho_n\infty}(x_n,\sigma(s)).\dot\sigma(s) = \partial_v L_Z(\sigma(s),\dot\sigma(s)).\dot\sigma(s)\\ &= \langle \dot\sigma(s) - Z(\sigma(s)), \dot\sigma(s)\rangle\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ is the scalar product associated to the norm $\|\cdot\|$. We deduce from these identities that

$$\langle \dot\sigma(s) - Z(\sigma(s)), \dot\sigma(s)\rangle = \frac{1}{2}\|\dot\sigma(s) - Z(\sigma(s))\|^2 = \left\langle \dot\sigma(s) - Z(\sigma(s)), \frac{1}{2}\dot\sigma(s) - \frac{1}{2}Z(\sigma(s))\right\rangle$$

and

$$0 = \langle \dot\sigma(s) - Z(\sigma(s)), \dot\sigma(s) + Z(\sigma(s))\rangle = \|\dot\sigma(s)\|^2 - \|Z(\sigma(s))\|^2$$

We obtain

$$|\dot\sigma_\delta(s)| = \|\dot\sigma(s)\| = \|Z(\sigma(s))\| = |Z_\delta(\sigma(s))|$$

The vector field $Z$ is directed from $\partial B_n^0$ towards $x_n$, so that $Z_\delta$ is negative on $B_n^0 \setminus \{x_n\}$. In particular, it is non-null and we deduce that either $\dot\sigma_\delta(s) = Z_\delta(\sigma(s))$ for all $s < t$, or $\dot\sigma_\delta(s) = -Z_\delta(\sigma(s))$ for all $s < t$. However, we have $\alpha_1(\sigma) = \{x_n\}$. Thus, for all $s < t$

$$\dot\sigma_\delta(s) = -Z_\delta(\sigma(s)) \quad \text{and} \quad \dot\sigma(s) = -Z(\sigma(s))$$

which extends by continuity to $s = t$. Since $\sigma(t) = y$, we conclude that for all $s \leq t$,

$$\sigma(s) = \phi_{-Z}^{s-t}(y) = \phi_{-Z}^{s-t} \circ \phi_{-Z}^{t}(x) = \phi_{-Z}^{s}(x) = \sigma_x(s)$$

We proved that $\sigma_x$ is calibrated on $(-\infty, t]$ for arbitrary. Hence, it is calibrated on $\mathbb{R}$. □

We conclude this section with the...

*Proof of Lemma 8.3.6.* Set $x^{\pm}$ to be the points of $A_n$ written in the coordinates $(r, \theta, x_3, .., x_d)$ as $x^{\pm} = (r_n \pm \frac{3\delta_n}{2}, 0, .., 0)$. We consider the curves

$$\sigma^{\pm}(t) = \phi_{-Z}^{t}(x^{\pm}) \quad \text{and} \quad \gamma^{\pm}(t) = \mathcal{R}_t \circ \sigma^{\pm}(t) \tag{8.3.19}$$

From the dynamics of $-Z$, we infer that $\alpha(\sigma^{\pm}) = \{z_n^{\pm}\}$ and $\omega(\sigma^{\pm}) \subset \partial B_n$. And by Lemmas 8.3.14 and 8.3.12, we deduce that $\sigma^{\pm}$ is calibrated by $h_Z^{\infty}(z_n^{\pm}, \cdot)$ and

$$\begin{aligned} h_Z^{\infty}(z_n^{\pm}, \partial B_n) &= \lim_{\substack{s \to -\infty \\ t \to +\infty}} h_Z^{s,t}(\sigma^{\pm}(s), \sigma^{\pm}(t)) = \int_{\mathbb{R}} L_Z(\sigma^{\pm}(\tau), \dot{\sigma}^{\pm}(\tau))\, d\tau \\ &= \int_{\mathbb{R}} \frac{1}{2} \|\dot{\sigma}^{\pm}(\tau) - Z(\sigma^{\pm}(\tau))\|^2\, d\tau = \int_{\mathbb{R}} \|Z(\sigma^{\pm}(\tau))\|^2\, d\tau \end{aligned} \tag{8.3.20}$$

Let us show that $\|Z(\sigma^{\pm}(\tau))\| = \|\dot{\sigma}^{\pm}(\tau)\|$. We have by definition of the curves $\sigma^{\pm}$ that

$$\begin{cases} \dot{\sigma}^{\pm}(\tau) = -Z(\sigma^{\pm}(\tau)) \\ \sigma^{\pm}(0) = x^{\pm} \end{cases}$$

We adopt the notation of Lemma 8.3.12 and we consider the coordinate $\delta$ defined by $x_\delta = d(O_n, x)$. The symmetries of the vector field $Z$ stated in Remark 8.1.5 show that $Z_\delta = \|Z\|$ and that it only depends on $x_\delta$ for $x \in A_n$. Hence, we deduce that $\sigma_\delta^{\pm}$ both verify the ODE

$$\begin{cases} \dot{\sigma}_\delta^{\pm}(\tau) = -Z_\delta(\sigma_\delta^{\pm}(\tau)) \\ \sigma_\delta^{\pm}(0) = \frac{3\delta_n}{2} \end{cases}$$

which has a unique solution. This yields the equalities $\sigma_\delta^{+} = \sigma_\delta^{-}$ and $\dot{\sigma}_\delta^{+} = \dot{\sigma}_\delta^{-}$. Going back to (8.3.20), we conclude that

$$h_Z^{\infty}(z_n^{+}, \partial B_n) = \int_{\mathbb{R}} \|Z(\sigma^{+}(\tau))\|^2\, d\tau = \int_{\mathbb{R}} \|Z(\sigma^{-}(\tau))\|^2\, d\tau = h_Z^{\infty}(z_n^{-}, \partial B_n)$$

In order to prove the equality for the Peierls barrier $h^{\infty}$ associated to $L$, we use Proposition 8.3.10 which shows that the curves $\gamma^{\pm}$ defined in (8.3.19) are calibrated by $h^{\infty}(z_n^{\pm}, \cdot)$

and we use the analogous of identity (8.3.14) to $h^\infty(z_n^\pm, \cdot)$ which gives

$$h_Z^\infty(z_n^\pm, \partial B_n) = \lim_{t\to+\infty} h_Z^{\infty+t}(z_n^\pm, \sigma^\pm(t)) = \lim_{t\to+\infty} h^{\infty+t}(z_n^\pm, \gamma^\pm(t)) = h^\infty(z_n^\pm, \partial B_n)$$

□

**Proof of Theorem 8.0.1**

We establish the $C^\infty$ regularity of $u_c(t,x)$. For that purpose, we define for all subset $F$ of $M$, the subset $RF$ of $\mathbb{R}\times M$ given by

$$RF := \cup_{s\in\mathbb{R}}\{s\}\times\mathcal{R}_z(F) \tag{8.3.21}$$

Hence, we obtain

$$\begin{aligned} RB_n^0 &= \bigcup_{s\in\mathbb{R}}\{s\}\times\mathcal{R}_s(B_n^0) \\ R(B_n^0\smallsetminus\{x_n\}) &= \bigcup_{s\in\mathbb{R}}\{s\}\times\mathcal{R}_s(B_n^0\smallsetminus\{x_n\}) \\ RB_n &= \bigcup_{s\in\mathbb{R}}\{s\}\times\mathcal{R}_s(B_n) \\ RA_n &= \mathbb{R}\times A_n \\ RC_n &= \mathbb{R}\times C_n \\ RD &= \mathbb{R}\times D \end{aligned} \tag{8.3.22}$$

**Proposition 8.3.15.** *The restriction of the solution $u_c$ to the closure of $RB_n^0$ is $C^\infty$ regular and all its derivatives are null on the boundary $\partial R(B_n^0\smallsetminus\{x_n\})$.*

*Proof.* Fix an integer $n\geq 0$. We first focus on the set $R(B_n^0\smallsetminus\{x_n\})$. The proof strategy involves constructing a foliation of $R(B_n^0\smallsetminus\{x_n\})$ by calibrated curves. According to Proposition 8.3.2, in this set, $u_c(t,x) = c_n + h^{\rho_n\infty+t}(x_n, x)$. Thus, we will aim to prove the calibration for the barrier $h^{\rho_n\infty+t}(x_n,\cdot)$ on the set $\mathcal{R}_t(B_n^0)$.

Let $(t,x)$ be in $R(B_n^0\smallsetminus\{x_n\})$. We set the curves $\sigma_y$ and $\gamma_{(t,x)}:\mathbb{R}\to M$ defined by

$$y = \phi_{-Z}^{-t}\circ\mathcal{R}_t^{-1}(x), \quad \sigma_y(\tau) = \phi_{-Z}^{\tau}(y) \quad \text{and} \quad \gamma_{(t,x)}(\tau) = \mathcal{R}_\tau\circ\sigma_y(\tau) \tag{8.3.23}$$

Lemma 8.3.14 states that $\sigma_y$ is calibrated by $h_Z^{\rho_n\infty}(x_n,\cdot)$. Hence, Lemma 8.3.10 states that $\gamma_{(t,x)}$ is calibrated by $h^{\rho_n\infty}(x_n,\cdot)$. Moreover, we have

$$\gamma_{(t,x)}(t) = \mathcal{R}_t\circ\sigma_y(t) = \mathcal{R}_t\circ\phi_{-Z}^{t}(y) = \mathcal{R}_t\circ\phi_{-Z}^{t}\circ\phi_{-Z}^{-t}\circ\mathcal{R}_t^{-1}(x) = x$$

Hence, we deduce from Theorem 5.1.20 that $u_c$ is differentiable at $(t,x)$ and

$$\begin{aligned} du_c(t,x) &= (\partial_t u_c(t,x), d_x u_c(t,x)) = \big( -H(t,x,d_x u_c(t,x)), \partial_v L(t,x,\dot\gamma_{(t,x)}(t))\big) \\ &= \left(-H(t,x,d_x u_c(t,x)), \partial_v L\left(t,x, \frac{d}{d\tau}\Big|_{\tau=t} \left(\mathcal{R}_\tau \circ \phi_{-Z}^{\tau-t} \circ \mathcal{R}_t^{-1}(x)\right)\right)\right) \end{aligned} \tag{8.3.24}$$

Therefore, $u_c$ is $C^\infty$ regular on $R(B_n^0 \smallsetminus \{x_n\})$.

This formula extends to the closure of $R(B_n^0 \smallsetminus \{x_n\})$. Indeed, we have $Z_{|\partial R(B_n^0\smallsetminus\{x_n\})} = 0$ and for all $(t,x) \in \partial R(B_n^0 \smallsetminus \{x_n\})$, the curve $f_{t,\tau}(x) = \mathcal{R}_{t,\tau}(x)$ is calibrated and of null action. Hence, the restriction of $u_c$ to the closure of $R(B_n^0 \smallsetminus \{x_n\})$ is of $C^\infty$ regularity. And in particular, it is $C^\infty$ on $RB_n^0$ and in particular, at $(t, \mathcal{R}_t(x_n))$.

Let us compute its derivative on the boundary $\partial RB_n^0$. For all $(t,x) \in RB_n^0$, if we set $y = \phi_{-Z}^{-t} \circ \mathcal{R}_t^{-1}(x)$, we have

$$\begin{aligned} \frac{d}{d\tau}\Big|_{\tau=t} \left(\mathcal{R}_\tau \circ \phi_{-Z}^{\tau}(y)\right) &= \frac{d}{dt}\mathcal{R}_t \circ \phi_{-Z}^t(y) + d\mathcal{R}_t.\frac{d}{dt}\phi_{-Z}^t(y) \\ &= \frac{d}{dt}\mathcal{R}_t \circ \mathcal{R}_t^{-1}(x) - d\mathcal{R}_t.Z\mathcal{R}_t^{-1}(x) \\ &= Y_t(x) - d\mathcal{R}_t.Z\mathcal{R}_t^{-1}(x) \end{aligned}$$

where $Y_t$ is the vector field associated to the isotopy $\mathcal{R}_t$. Additionally, we know from (8.1.13) and (8.1.14) that

$$\partial_v L(t,x,v) = v - X_t(x) = v - Y_t(x) - d\mathcal{R}_t.Z\mathcal{R}_t^{-1}(x)$$

Thus, we obtain

$$\begin{aligned} d_x u_c(t,x) &= \partial_v L\left(t,x, \frac{d}{d\tau}\Big|_{\tau=t} \left(\mathcal{R}_\tau \circ \phi_{-Z}^{\tau-t} \circ \mathcal{R}_t^{-1}(x)\right)\right) \\ &= Y_t(x) - d\mathcal{R}_t.Z\mathcal{R}_t^{-1}(x) - Y_t(x) - d\mathcal{R}_t.Z\mathcal{R}_t^{-1}(x) = -2d\mathcal{R}_t.Z\mathcal{R}_t^{-1}(x) \end{aligned}$$

Moreover, we have the limits

$$\lim_{(t,x)\to\partial RB_n^0} d(B_n^0, \mathcal{R}_t^{-1}(x)) = 0 \quad \text{and} \quad \lim_{y\to B_n^0} Z = 0$$

where the last limit on $Z$ is in the $C^\infty$-topology. Therefore, we deduce that $d_x u_c$ converges to 0 in the $C^\infty$ topology as $(t,x)$ converges to $\partial B_n^0$.

We also compute $\partial_t u_c$ using the expression (8.1.15) of the Hamiltonian $H$. This yields

$$\begin{aligned} \partial_t u_c(t,x) &= -H(t,x,d_x u_c(t,x)) = -H(t,x,-2d\mathcal{R}_t.Z\mathcal{R}_t^{-1}(x)) \\ &= \frac{1}{2}\| - 2d\mathcal{R}_t.Z\mathcal{R}_t^{-1}(x) + X_t(x)\|^2 - \frac{1}{2}\|X_t(x)\|^2 \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\|Y_t(x) - d\mathcal{R}_t.Z\mathcal{R}_t^{-1}(x)\|^2 - \frac{1}{2}\|Y_t(x) - d\mathcal{R}_t.Z\mathcal{R}_t^{-1}(x)\|^2 \\
&= -\langle Y_t(x), d\mathcal{R}_t.Z\mathcal{R}_t^{-1}(x)\rangle
\end{aligned}
$$

which also converges to 0 in the $C^\infty$ topology as $(t,x)$ goes to $\partial RB_n^0$. We proved that $du_c$ converges to 0 in the $C^\infty$ topology as $(t,x) \in RB_n^0$ goes to the boundary $\partial RB_n^0$. □

**Proposition 8.3.16.** *The restriction of the solution $u_c$ to the closure of $RA_n := \mathbb{R} \times A_n$ is $C^\infty$ regular and all its derivatives are null on the boundary $\partial RB_n^0$.*

*Proof.* In the 2D case, we have shown in Proposition 8.3.2 that $u_c$ is constant on $RA_n$ which implies the result.

In the 3D case, the same Proposition 8.3.2 asserts that for all $(t,x) \in RA_n$, $u(t,x) = h^{\infty+t}(z_\infty, x)$. Applying the second property of Lemma 8.3.14, we prove analogously to Proposition 8.3.15 that the curves $\gamma_{t,x}(\tau) = \mathcal{R}_\tau \circ \phi_{-Z}^{\tau-t} \circ \mathcal{R}_t^{-1}(x)$ are calibrated by $u_c$, and we obtain the same formula (8.3.24) on $du_c$ which also converges to 0 at the boundary $\partial RA_n$ of $RA_n$ in the $C^\infty$ topology. □

*Proof of Theorem 8.0.1.* Gathering the results of Propositions 8.3.2, 8.3.15 and 8.3.16, we have shown that $u_c$ is $C^\infty$ regular on the set

$$
RD \cup \bigcup_{n\geq 0}\left(RA_n \cup RB_n^0 \cup (RB_n \setminus RB_n^0)\right) = M \setminus \left(\bigcup_{n\geq 0} \partial RC_n \cup \partial RB_n \cup \partial RB_n^0\right)
$$

In order to complete the proof of Theorem 8.0.1, it suffices to prove $C^\infty$ regularity on $\partial RC_n$, $\partial RB_n$ and $\partial RB_n^0$.

*On $\partial RC_n$.* We have that $u_c$ is locally constant on $RD$ and all the derivatives of $du_{c|RA_n}$ converge to 0 at the external boundary $\partial RC_n$ of $RA_n$. Hence, $u_c$ is $C^\infty$ at $\partial RC_n$ with null derivatives.

*On $\partial RB_n^0$ and $\partial RB_n$.* We have that $u_c$ is constant on $RB_n \setminus RB_n^0$, and all the derivatives of $du_{c|RA_n}$ and $du_{c|RB_n^0}$ converge to 0 at their boundary, and in particular at $\partial RB_n^0$ and $\partial RB_n$. Hence, $u_c$ is $C^\infty$ at $\partial RB_n^0$ and $\partial RB_n$ with null derivatives.

This concludes the proof of the main Theorem 8.0.1. □

**Remark 8.3.17.** 1. Lemma 8.3.14 can be understood as follows : The Mañé set $\tilde{\mathcal{N}}_Z$ corresponding to the autonomous component $Z$ of $X_t$ is symmetric with respect to the zero section of $TM$. The radial symmetry imposed on $Z$ yields a Mañé set that, when viewed radially, closely resembles what is observed in one-dimensional systems, specifically the phase portrait of a pendulum. A radial section of $\tilde{\mathcal{N}}_Z$ around $r_n$ is represented in Figure 8.3.

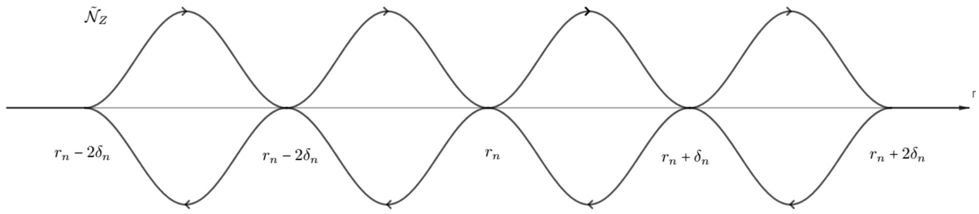



FIGURE 8.3 – Radial Section at angle $\theta = 0$ of the Mañé Set $\tilde{\mathcal{N}}_Z$.

2. Note that the obtained $C^1$ regularity is highly unstable. A generic perturbation of the constants $c_n$ induces irregularities. Specifically, if the equality $c_n + h^{\rho_n \infty}(x_n, x) = h^\infty(z_\infty, x)$ occurs outside the Peierls set $\mathcal{A}_0$, then the loss of regularity is inevitable. In other words, the set of $C^1$ recurrent viscosity solutions is sparse within the non-wandering set $\Omega(\mathcal{T})$ of the Lax-Oleinik operator $\mathcal{T}$.
3. Similarly, any perturbation of the Hamiltonian that disrupts its symmetries may also result in a loss of regularity, and such a Hamiltonian may not possess any $C^1$ recurrent viscosity solution. There are examples of Hamiltonian systems with no regular elements in their non-wandering set $\Omega(\mathcal{T})$. The simplest example is the autonomous simple pendulum, where the unique element of $\Omega(\mathcal{T})$ is a Lipschitz weak-KAM solution.
4. Additionally, obtaining $C^{1,1}$ recurrent viscosity solutions is significantly easier than obtaining $C^2$ or more regular solutions. This is because if the infimum of two Peierls barriers occurs on the Peierls set, $C^{1,1}$ regularity is guaranteed. Achieving higher regularity requires modifying the Hamiltonian to ensure that convergence to such points occurs only via parabolic trajectories.
5. From a symplectic perspective, the smooth recurrent locally viscosity solution $u_c$ gives rise to a Lagrangian submanifold, namely the graph of $du_c$ in the cotangent bundle $T^*M$, which is ($C^1$-) recurrent under the Hamiltonian flow $\phi_H$.

# Chapitre 9

# A Multidimensional Birkhoff Theorem for Recurrent Lagrangian Submanifolds by a Tonelli Hamiltonian

A theorem attributed to G. D. Birkhoff [Bir22] establishes that any non contractible invariant curve, which is preserved under an exact twist map of the annulus, is a Lipschitz graph over the circle. This theorem has inspired numerous efforts to establish various modern proofs. We can refer for example to [Her83, KO97, Sib98].

Since then, attempts were made to extend the result to higher dimensions. Under certain assumptions, the authors have managed to demonstrate that for a convex Hamiltonian of a cotangent bundle or a multidimensional positive twist map, an invariant exact Lagrangian submanifold is a graph.

For results on the multidimensional torus $\mathbb{T}^n$ with various conditions on the invariant Lagrangian, we can cite the works of Herman [Her89], Bialy and Polterovich [BP92], and Carneiro and Ruggiero [DCR23]. On general manifolds $M$, Arnaud [Arn10] proved that if a Lagrangian submanifold of $T^*M$, which is Hamiltonianly isotopic to the zero section, is fixed by an autonomous Tonelli Hamiltonian map, then it is a Lipschitz graph over the base manifold $M$. Later, Bernard and dos Santos [BdS12] extended this result to the case of Lipschitz Lagrangian submanifolds, and Amorim, Oh, and dos Santos [AOdS18] dropped the Hamiltonian isotopy condition, proving the theorem for exact-Lipschitz Lagrangian submanifolds using generalized graph selectors based on Floer theory.

In the non-autonomous setting, Arnaud and Venturelli [AV17] showed that the result of [Arn10] still holds. It is established that the periodic Lagrangian submanifold is the graph of some $du$, where $u$ is a weak-KAM solution of the Hamilton-Jacobi equation. This finding has been the starting point of the result of this chapter, suggesting a correspondence between larger sets of Lagrangian submanifolds (Hamiltonianly isotopic the the zero section) and solutions of the Hamilton-Jacobi equation.

Periodic Lagrangian submanifolds are replaced by Lagrangian submanifolds $\mathcal{L}$ which images under the action of a Tonelli Hamiltonian flow $\phi_H$ have convergent subsequences in both positive and negative times, for a type of convergence called reduced complexity convergence (see Definition 9.1.2). In other words, we require that two subsequences of $\phi_H^n(\mathcal{L})$ and of $\phi_H^{-n}(\mathcal{L})$ converge in the Hausdorff topology to another Lagrangian submanifold, with a control on the Liouville primitive to prevent winding phenomena.

In addition, We have seen in Theorem 6.0.2 that bounded global viscosity solutions of the Hamilton-Jacobi equation are recurrent and share the same properties as weak-KAM solutions, particularly those used to prove the Birkhoff theorem in [AV17].

As a result, it is established in this chapter that Lagrangian submanifolds that converge with reduced complexity in both positive and negative times to some limit Lagrangian subamnifolds are graphs of $du$, where $u$ is a bounded viscosity solution of the Hamilton-Jacobi equation. Consequently, it is shown that these submanifolds are recurrent under the action of the Hamiltonian flow, meaning that $\phi_H^n(\mathcal{L})$ has a subsequence that converges, with reduced complexity, in both positive and negative times, to the initial Lagrangian $\mathcal{L}$.

One difficulty is to define the appropriate topology for the recurrence of Lagrangian submanifolds in order to obtain the desired result. If the chosen topology is too lenient, it might permit the Lagrangians to wind around their limit, leading to a failure of the graph property. It will be shown that controlling the stretching of the submanifolds will suffice. See Definition 9.1.2 and Theorem 9.1.5.

Another difficulty is the construction of concrete examples of recurrent Lagrangian submanifolds under Tonelli Hamiltonian maps. This complicates the construction of counterexamples for loose topologies where recurrence does not necessarily imply a Birkhoff result. However, we know from Chapter 8 that the framework of this chapter is non-empty, as we constructed an example of a recurrent, non-periodic $C^1$ Lagrangian submanifold that is the graph of a recurrent viscosity solution.

To prove the main result, we rely heavily on the weak KAM theory, which was developed by A. Fathi in the 1990s [Fat97b, Fat08]. However, it is important to note that we deliberately refrained from relying on the entire theory, including concepts like the Aubry sets and weak KAM solutions. This deliberate choice ensures that our article stands as a self-contained work. An exception will be made for Corollary 9.1.9, the proof of which requires the properties of the Lax-Oleinik operators briefly discussed in Appendix B.

Additionally, a crucial element of our proof involves the use of a graph selector. These allow us to choose a pseudograph (a type of discontinuous exact Lagrangian graph) within the initial Lagrangian submanifold. The concept of graph selectors was first introduced by M. Chaperon [Cha91, PPS03]. For our purposes, we will adopt the construction given by C. Viterbo in [Vit96] relying on spectral invariants defined from generating functions (see [Vit92] or [Hum08] Chapter 1). For the sake of completeness, we have chosen to reprove the vast majority of the properties of these invariants by adapting them to the framework of this chapter.

## 9.1 Notations and Main Result

Fix a closed manifold $M$ of dimension $d$ endowed with a Riemannian metric $d$ and its relative norm $\|.\|$ on the cotangent bundle. The cotangent bundle $T^*M$ has a natural exact symplectic structure $(T^*M, \omega = -d\lambda)$ defined as follows. Let $\pi_M : T^*M \to M$ be the natural projection and denote by $(q,p) = (q_1,..,q_d,p_1,..,p_d)$ the coordinates in $T^*M$ where $q = (q_1,..,q_d)$ are local coordinates of $M$ and $p = (p_1,..,p_d)$ are fiberwise coordinates with respect to the cotangent vectors $dq_1,..,dq_d$. The *Liouville form* $\lambda$ is defined by $\lambda(q,p) = p \circ d\pi_M = p.dq$.

A *Lagrangian submanifold* $\mathcal{L}$ of $T^*M$ is a submanifold such that $\dim \mathcal{L} = d$ and $\omega_{|T\mathcal{L}} = 0$. The Lagrangian submanifold $\mathcal{L}$ is *exact* if $\lambda_{|T\mathcal{L}}$ is an exact form i.e there exist a *Liouville primitive* $h : \mathcal{L} \to \mathbb{R}$ such that $\lambda_{|\mathcal{L}} = dh$. We define the *oscillation* of $h$ as

$$\operatorname{Osc}(h) = \max h - \min h \tag{9.1.1}$$

The *Hausdorff distance* $d_H$ on the set of compact Lagrangian submanifolds of $T^*M$ is defined for all such $\mathcal{L}$ and $\mathcal{L}'$ as

$$d_H(\mathcal{L},\mathcal{L}') = \max\left\{ \sup_{x'\in\mathcal{L}'} d(x',\mathcal{L}) \ , \ \sup_{x\in\mathcal{L}} d(x,\mathcal{L}')\right\} \tag{9.1.2}$$

Set $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ and denote by $t$ the time coordinate in $\mathbb{T}^1$. A $C^2$ time-periodic *Hamiltonian* is a map $H : \mathbb{T}^1 \times T^*M \to \mathbb{R}$. Given $H$, the *Hamiltonian vector field* $X_H$ is defined by the equation $\iota_{X_H^t}\omega = \omega(X_H^t, \cdot) = dH_t$ where $H_t = H(t,\cdot,\cdot)$ and the corresponding *Hamiltonian flow* is denoted by $\phi_H^{s,t}$. We set $\phi_H^t := \phi_H^{0,t}$. The *Hamiltonian group* $\operatorname{Ham}(T^*M,\omega)$ is the group of Hamiltonian maps i.e time one of Hamiltonian flows.

**Definition 9.1.1.** A $C^2$ time-periodic Hamiltonian $H(t,q,p) : \mathbb{T}^1 \times T^*M \to \mathbb{R}$ is called *Tonelli* if it satisfies the following classical hypotheses :

— (*Strict convexity*) $\partial_{pp}H(t,q,p) > 0$ for all $(t,q,p) \in \mathbb{T}^1 \times T^*M$.

— (*Superlinearity*) $\frac{|H(t,q,p)|}{\|p\|} \to \infty$ as $\|p\| \to \infty$ for each $(t,q) \in \mathbb{T}^1 \times M$.

— (*Completeness*) The Hamiltonian vector field $X_H$ and hence its flow $\phi_H^{s,t}$ are complete in the sense that the flow curves are defined for all times $t \in \mathbb{R}$.

Tonelli Hamiltonians are the good setting to have a correspondence between Hamiltonian and Lagrangian dynamics. This is precisely the framework of Fathi's weak KAM theory. A flavour of this will be given in Section 9.5 and Appendix B. For a more detailed exposition on the subject, we refer to [Fat08].

We now fix a Lagrangian submanifold $\mathcal{L}$ that is *Hamiltonianly isotopic* or *H-isotopic* to the zero section $0_{T^*M}$, that is, there exists a Hamiltonian map $\varphi \in \mathrm{Ham}(T^*M,\omega)$ such that $\mathcal{L} = \varphi(0_{T^*M})$. For all time $t$ in $\mathbb{R}$, we set $\mathcal{L}_t = \phi_H^t(\mathcal{L})$ and $\mathcal{L}'_t = \varphi^{-1}(\mathcal{L}_t)$.

**Definition 9.1.2.** Let $(\mathcal{L}_n)_{n\geq 0}$ and $\mathcal{L}$ be a exact Lagrangian submanifolds of $T^*M$ $H$-isotopic to the null section $0_{T^*M}$. We say that the sequence $(\mathcal{L}_n)_{n\geq 0}$ *converges with reduced complexity* if

i. $\lim_n d_H(\mathcal{L}_n, \mathcal{L}) = 0$.

ii. For a Hamiltonian map $\varphi$ such that $\mathcal{L} = \varphi(0_{T^*M})$, if $l_n$ is a Liouville primitive on the Lagrangian submanifold $\varphi^{-1}(\mathcal{L}_n)$, then $\lim_n \mathrm{Osc}(l_n) = 0$.

**Proposition 9.1.3.** *The definition of reduced complexity convergence does not depend on the choice of the Hamiltonian map $\varphi$ such that $\mathcal{L} = \varphi(0_{T^*M})$.*

**Remark 9.1.4.**

1. The idea of this convergence is to constraint the winding of the Lagrangian submanifolds around the limit points. More precisely, if $h_n$ and $h_\omega$ are Liouville primitives on $\mathcal{L}_n$ and $\mathcal{L}_\omega$, we aim to control the oscillation of "$h_n - h_\omega$". However, the two primitives have different definition domains and we need to make a change of variable in order to reduce $h_\omega$ into a constant primitive $l_\omega$ of $\varphi_\omega^{-1}(\mathcal{L}_\omega) = 0_{T^*M}$.
2. Comparing the definition to other types of convergence, we have
   (a) The $C^1$ convergence of $(\mathcal{L}_n)_n$ to $\mathcal{L}_\omega$ implies that the sequence is of reduced asymptotic complexity.
   (b) Reduced asymptotic complexity implies spectral convergence by the mean of the spectral distance $\gamma$ introduced by C.Viterbo in [Vit92]. More precisely, if $(\mathcal{L}_n)_{n\geq 0}$ is of reduced asymptotic complexity and converges to $\mathcal{L}_\omega$, then $\lim_n \gamma(\mathcal{L}_n, \mathcal{L}_\omega) = 0$. However the converse is false since spectral convergence allows non-boundedness of $\mathrm{Osc}(l_n)$.

We state the main theorem of the chapter.

**Theorem 9.1.5.** *Let $M$ be a closed manifold, $H:\mathbb{T}^1\times T^*M\to\mathbb{R}$ be a Tonelli Hamiltonian with flow $\phi_H$ and let $\mathcal{L}$ be a Lagrangian submanifold of $T^*M$ which is $H$-isotopic to the zero section. For all time $t\in\mathbb{R}$, we set $\mathcal{L}_t=\phi_H^t(\mathcal{L})$. Suppose that there exist two Lagrangian submanifolds $\mathcal{L}_\omega$ and $\mathcal{L}_\alpha$ $H$-isotopic to the zero section, and two increasing sequences of integers $n_k$ and $m_k$ such that $(\mathcal{L}_{n_k})_{k\geq 0}$ and $(\mathcal{L}_{-m_k})_{k\geq 0}$ converge with reduced complexity to $\mathcal{L}_\omega$ and $\mathcal{L}_\alpha$ respectively. Then $\mathcal{L}$ and all its images $\mathcal{L}_t$ are $C^1$ graphs over the zero section $0_{T^*M}$ of $T^*M$.*

In particular, this Theorem applies for reccurent Lagrangian submanifolds with a control on the Liouville primitives.

**Corollary 9.1.6.** *Let $M$ be a closed manifold, $H:\mathbb{T}^1\times T^*M\to\mathbb{R}$ be a Tonelli Hamiltonian with flow $\phi_H$ and let $\mathcal{L}$ be a Lagrangian submanifold of $T^*M$ which is $H$-isotopic to the zero section. If $\mathcal{L}$ is a positively and negatively time recurrent Lagrangian for the reduced complexity convergence, meaning that if there exist increasing sequences of integers $n_k$ and $m_k$ such that $(\mathcal{L}_{n_k})_k$ and $(\mathcal{L}_{-m_k})_k$ converge with reduced complexity to $\mathcal{L}$, then $\mathcal{L}$ and all its images $\mathcal{L}_t$ are $C^1$ graphs over the zero section $0_{T^*M}$ of $T^*M$.*

An immediate consequence is the result by Marie-Claude Arnaud and Andrea Venturelli [AV17] on periodic Lagrangian submanifolds under Tonelli Hamiltonian maps.

**Corollary 9.1.7.** *Let $M$ be a closed manifold. If a Lagrangian submanifold $\mathcal{L}$ of $T^*M$ is periodic under the Hamiltonian map $\phi_H^1$ of a Tonelli Hamiltonian $H$, then $\mathcal{L}$ is a $C^1$ graph over the zero section $0_{T^*M}$ of $T^*M$.*

It is possible to obtain more information about the scalar map $u:\mathbb{R}\times M\to\mathbb{R}$ for which $\mathcal{L}_t$ is the graph of $du(t,\cdot)$. Further study, focused on weak-KAM theory and viscosity solutions theory, reveals that $u(t,q)$ is a recurrent (viscosity) solution of the Hamilton-Jacobi equation

$$\partial_t u+H(t,q,d_q u(t,q))=\alpha_0 \tag{9.1.3}$$

where $\alpha_0$ is the Mañé critical value (see Identity 5.1.6 in Definition 5.1.3). This is contained in the establishment of following.

**Corollary 9.1.8.** *Under the assumptions of Theorem 9.1.5, if there exist two increasing integer sequences $n_k$ and $m_k$ such that $(\mathcal{L}_{n_k})_{k\geq 0}$ and $(\mathcal{L}_{-m_k})_{k\geq 0}$ have reduced asymptotic complexity, then $\mathcal{L}$ is a $\phi_H^1$-recurrent Lagrangian submanifold for the Hausdorff distance.*

**Corollary 9.1.9.** *Under the assumptions of Theorem 9.1.5, the Lagrangian submanifold $\mathcal{L}$ is a $\phi_H^1$-recurrent Lagrangian submanifold for the Hausdorff distance.*

In the autonomous case, A.Fathi ([Fat98], Corollary 7.1.3) established a convergence theorem which tells that recurrent (viscosity) solutions of the Hamilton-Jacobi equation are stationary, meaning that they are time-independent and thus satisfy the equation

$$H(q, d_q u(t,q)) = \alpha_0 \tag{9.1.4}$$

As a consequence, we derive the following autonomous version of Corollary 9.1.9.

**Corollary 9.1.10.** *Under the assumptions of Theorem 9.1.5 and if $H : T^*M \to \mathbb{R}$ is autonomous, the Lagrangian submanifold $\mathcal{L}$ is a $C^1$ graph over the zero section $0_{T^*M}$, and it is a $\phi_H$-invariant Lagrangian submanifold i.e for all time $t \in \mathbb{R}$, $\phi_H^t(\mathcal{L}) = \mathcal{L}$.*

**Remark 9.1.11.** Many questions arise about possible generalizations :

1. (Types of recurrence) The reduced complexity convergence may be considered too restrictive. The proof presented here works when the variation between the Liouville primitives on $\mathcal{L}_{-m_k}$ and $\mathcal{L}_{n_k}$ converges to zero as this provides calibration for limit curves that will allow the use of results from Fathi's weak-KAM theory. The conclusion we obtain is that $\mathcal{L}$ is the graph of $du$ where $u$ is a (viscosity) solution of the Hamilton-Jacobi equation.
   However, this reasoning fails when this variation between the Liouville primitives grows as it may happen if we only assume the Hausdorff convergence. In this case, we failed to make a working proof nor could we provide a counter-example.

   **Question 9.1.12.** Is a Birkhoff theorem still valid if we only assume a Hausdorff convergence, or more reasonably, spectral convergence of the sequences $\mathcal{L}_{-m_k}$ and $\mathcal{L}_{n_k}$ respectively to $\mathcal{L}_\alpha$ and $\mathcal{L}_\omega$. Here, spectral convergence stands for Viterbo's gamma distance $\gamma$ on Lagrangian submanifolds.

2. (Negative times) It seems that for the case of viscosity solutions $u(t,x)$, negative time-recurrence implies positive-time recurrence (see Appendix B). Therefore, this is the case of the Lagrangian $\mathcal{L} = \mathcal{G}(d_q u(0,\cdot))$. And since the Lagrangian submanifolds considered in Theorem 9.1.5 turn out to be graphs of the differential of recurrent viscosity solutions, it feels natural to ask the question.

   **Question 9.1.13.** Is the Birkhoff theorem still true if we only assume a negative-time reduced complexity convergence ? Alternatively, is it possible to construct a negative-time-recurrent Lagrangian submanifold $\mathcal{L}$ that is not recurrent in positive times ?

   Providing a counter-example turned out to be trickier than expected as examples of recurrent Lagrangian submanifolds are lacking to the literature.

*Proof of Proposition 9.1.3.* Let $\varphi$ and $\psi$ be two Hamiltonian maps such that $\mathcal{L} = \varphi(0_{T^*M}) = \psi(0_{T^*M})$. We set $k_n$ to be Liouville primitives on the Lagrangian submanifolds $\varphi^{-1}(\mathcal{L}_n)$. We get Liouville primitives $l_n$ on $\psi^{-1}(\mathcal{L}_n)$ using the following lemma.

**Lemma 9.1.14.** *Let $f$ be an exact symplectomorphism of $T^*M$ and let $g : T^*M \to \mathbb{R}$ be a scalar map such that $f^*\lambda - \lambda = dg$. Then, if $\mathcal{L}$ is an exact Lagrangian submanifold of $T^*M$ with Liouville primitive $k$, then $f^{-1}(\mathcal{L})$ is an exact Lagrangian submanifold of $T^*M$ with Liouville primitive $l = k \circ f - g$.*

*Démonstration.* Let $x$ and $y$ be two points of $f^{-1}(\mathcal{L})$ linked by a curve $\gamma$ in $f^{-1}(\mathcal{L})$ and let $x' = f(x)$ and $y' = f(y)$ linked by $\gamma' = f(\gamma)$ in $\mathcal{L}$. We will evaluate $\int_\gamma \lambda$. We have

$$(f^{-1})^*\lambda = (f^{-1})^*(f^*\lambda - dg) = \lambda + d(-g \circ f^{-1})$$

We set $h := -g \circ f^{-1}$. Then we have

$$\int_\gamma \lambda = \int_{\gamma'} (f^{-1})^*\lambda = \int_{\gamma'} \lambda + dh = \int_{\gamma'} dk + dh = \int_{\gamma'} d(k+h) = \int_\gamma d\big((k+h)\circ f\big)$$

where

$$(k+h)\circ f = k \circ f - g = l$$

Hence, the Lagrangian submanifold $f^{-1}(\mathcal{L})$ is exact with Liouville primitive $l$. □

We apply the lemma for $f = \varphi^{-1} \circ \psi$ so that $\psi^{-1}(\mathcal{L}_n) = f^{-1}(\varphi^{-1}(\mathcal{L}_n))$ and we choose $l_n = k_n \circ f - g$ to be Liouville primitives on $\psi^{-1}(\mathcal{L}_n)$, where $f^*\lambda - \lambda = dg$. We need to prove that $\lim_n \operatorname{Osc}(l_n) = 0$. We have

$$\operatorname{Osc}(l_n) = \operatorname{Osc}(k_n \circ f - g) \leq \operatorname{Osc}(k_n \circ f) + \operatorname{Osc}(g_{|\psi^{-1}(\mathcal{L}_n)}) = \operatorname{Osc}(k_n) + \operatorname{Osc}(g_{|\psi^{-1}(\mathcal{L}_n)})$$

Since the reduced complexity convergence provides the first limit $\lim_n \operatorname{Osc}(k_n) = 0$, it suffices to show that $\lim_n \operatorname{Osc}(g_{|\psi^{-1}(\mathcal{L}_n)})$. We know that $\mathcal{L} = \varphi(0_{T^*M}) = \psi(0_{T^*M})$ so that $f(0_{T^*M}) = 0_{T^*M}$. This gives $f^*\lambda_{|0_{T^*M}} = 0$ and we get

$$dg_{|0_{T^*M}} = (f^*\lambda - \lambda)_{|0_{T^*M}} = 0$$

and consequently, $g$ is constant on the zero-section $0_{T^*M}$, or in other words $\operatorname{Osc}(g_{|0_{T^*M}}) = 0$. Now, we know by assumption that $\lim_n d_H(\mathcal{L}_n, \mathcal{L}) = 0$, or by continuity of the Hamiltonian flow $\psi$, that $\lim_n d_H(\psi^{-1}(\mathcal{L}_n), 0_{T^*M}) = 0$. Hence, uniform continuity of the map $g$ on a compact neighbourhood of the zero-section $0_{T^*M}$ yields the desired limit $\lim_n \operatorname{Osc}(g_{|\psi^{-1}(\mathcal{L}_n)})$. □

The Section 9.2 of this chapter is devoted to the definition of an extended autonomous Hamiltonian $\mathscr{H}$ and an extended Lagrangian submanifold $\mathscr{L}$ in the new phase space $T^*(\mathbb{R}\times M)$. In Section 9.3, we recall the main properties of generating functions and Liouville primitives and we construct convenient ones for $\mathscr{L}$. Section 9.4 contains a reminder on spectral invariants and graph selectors with the properties that will play a central role in the proof of the main theorem. A fitting graph selector $u$ of $\mathscr{L}$ will then be defined. The following Section 9.5 is dedicated to the proof of the main results of this chapter, using various concepts coming from the weak KAM Theory. An Appendix B introduces complementary tools that will serve to prove Corollary 9.1.9.

A knowledgeable reader about generating functions and spectral invariants can start with Section 9.2 and skip to Subsection 9.4.3 which deals with the construction of the extended graph selector.

## 9.2 Extension of the Lagrangian Submanifold

To have a global view over all the Lagrangian submanifolds $\mathcal{L}_t = \phi^t_H(\mathcal{L})$, we will consider time as part of the manifold variables and combine them into a single Lagrangian submanifold. Let $\mathcal{M} = \mathbb{R}\times M$ be a non-compact manifold with cotangent bundle $T^*\mathcal{M} = T^*\mathbb{R}\times T^*M$. We denote the coordinates in $T^*\mathcal{M}$ by $(\tau, E, q, p)$. The 1-form $\Lambda = \lambda + Ed\tau$ is a Liouville form for $T^*\mathcal{M}$ endowed with the symplectic form $\Omega = -d\Lambda = \omega + d\tau\wedge dE$

We extend the Hamiltonian $H : \mathbb{T}^1\times T^*M \to \mathbb{R}$ to a Hamiltonian $\mathscr{H} : T^*\mathbb{R}\times T^*M\to\mathbb{R}$ defined by

$$\mathscr{H}(\tau, E, q, p) = E + H(\tau, q, p) \tag{9.2.1}$$

And we extend the exact Lagrangian submanifolds $\mathcal{L}_t$ to a Lagrangian submanifold of $T^*\mathcal{M}$ defined by

$$\mathscr{L} := \left\{\phi^t_{\mathscr{H}}\big(0, -H(0,q,p), q, p\big) \mid (q,p)\in\mathcal{L},\ t\in\mathbb{R}\right\} \tag{9.2.2}$$

Note that for $(q,p)\in\mathcal{L}$, $\mathscr{H}\big(0,-H(0,q,p),q,p\big) = -H(0,q,p) + H(0,q,p) = 0$, and since the Hamiltonian is autonomous, we get the inclusion

$$\mathscr{L}\subset\{\mathscr{H} = 0\} \tag{9.2.3}$$

We will see in Proposition 9.3.7 that the Lagrangian submanifold $\mathscr{L}$ is exact.

## 9.3 Generating Functions

This section is devoted to a brief presentation of generating functions and their main properties. Although everything presented here is known, we will give proofs of all properties except for Sikorav's existence theorem and Viterbo's uniqueness theorem.

Moreover, convenient generating functions for $\mathscr{L}$ will be constructed along the exposition.

### 9.3.1 Generating Functions

Let $\mathcal{M}$ be a manifold, not necessarily compact. Let $p : E \to \mathcal{M}$ be a finite-dimensional vector bundle over $\mathcal{M}$.

**Definition 9.3.1.** A $\mathcal{C}^2$ map $S : E \to \mathbb{R}$ is a *generating function* if

i. Zero is a regular value of the map

$$\begin{aligned} E &\to T^*E \\ (q,\xi) &\mapsto d_\xi S(q,\xi) \end{aligned}$$

So that the *critical locus* $\Sigma_S = \{(q,\xi) \in E \mid d_\xi S(q,\xi) = 0\}$ is a submanifold of $E$.

ii. The map

$$\begin{aligned} i_S : \Sigma_S &\to T^*\mathcal{M} \\ (q,\xi) &\mapsto (q, d_q S(q,\xi)) \end{aligned} \tag{9.3.1}$$

is a diffeomorphism from $\Sigma_S$ onto its image $\mathcal{L}_S = i_S(\Sigma_S)$. $\mathcal{L}_S$ is the (exact) Lagrangian submanifold generated by $S$.

When the bundle $E$ is trivial i.e. $E = \mathcal{M} \times \mathbb{R}^k$ and there exists a non-degenerate quadratic form $Q : \mathbb{R}^k \to \mathbb{R}$ and a real constant $c \in \mathbb{R}$ such that $S = c + Q$ outside of a compact subset of $E$, we say that $S$ is a *generating function quadratic at infinity* or *g.f.q.i.* The *index* of $S$ is the index of the quadratic form $Q$.

**Theorem 9.3.2.** *(Sikorav's existence theorem [Sik87]) Let $\mathcal{L}$ be a Lagrangian submanifold of $T^*\mathcal{M}$ that admits a g.f.q.i and let $\phi^t$ be a Hamiltonian isotopy. Then there exists a smooth path of g.f.q.i $(S_t : M \times \mathbb{R}^k \to \mathbb{R})_{t\in[0,1]}$ with a fixed dimension $k$ such that for all $t \in [0,1]$, $S_t$ generates $\phi^t(\mathcal{L})$.*

This theorem proven in paragraph 1.7 of [Sik87] has been simplified by M.Brunella in [Bru91]. Actually, the method exposed in [Bru91] does not provide a smooth path, but gives only the generating function $S_1$. An easy adaptation proves this parametrized version (see [Hum08] Appendix B).

We can define for all time $T > 0$ the paths $(S_t^T : M \times \mathbb{R}^{k_T} \to \mathbb{R})_{t \in [-T,T]}$ such that $S_t^T$ generates $\mathcal{L}_t := \phi_H^t(\mathcal{L})$. We may sometimes write $S^T(t,x)$ instead of $S_t^T(x)$.

**Definition 9.3.3.** Let $S : E = \mathcal{M} \times \mathbb{R} \to \mathbb{R}$ be a generating function. The *basic operations* on generating functions are

- *(Translation)* $S' = S + c : E \to \mathbb{R}$ for some constant $c \in \mathbb{R}$.
- *(Bundle isomorphism)* $S' = S \circ F : E' \to \mathbb{R}$ for some bundle isomorphism $F : E' \to E$ where $p' : E' \to \mathcal{M}$ is a vector bundle over $\mathcal{M}$.
- *(Stabilization)* $S' = S \oplus Q' : E \oplus E' \to \mathbb{R}$ for some map $Q' : E' \to \mathbb{R}$ that is a non-degenerate quadratic form when restricted to the fibers of a finite dimensional vector bundle $p' : E' \to \mathcal{M}$.

Two generating functions $S$ and $S'$ are said *equivalent* if they can be made equal to a third generating function $S''$ after a succession of basic operations.

**Theorem 9.3.4.** *(Viterbo's uniqueness theorem [Vit92, Thé99]) If $S$ and $S'$ are two g.f.q.i that generate the same Lagrangian submanifold $\mathcal{L}$, then they are equivalent.*

### 9.3.2 Liouville Primitives

Recall that an exact Lagrangian submanifold $\mathcal{L}$ in $T^*M$ is a Lagrangian submanifold such that the Liouville form restricted to it $\lambda_{|\mathcal{L}}$ is exact i.e. $\lambda_{|\mathcal{L}} = dh$ with Liouville primitive $h : \mathcal{L} \to \mathbb{R}$.

The Lagrangian submanifolds $\mathcal{L}_t$ are exact and do admit Liouville primitives that can be deduced from their generating functions as follows

$$h_t^T = S_t^T \circ i_{S|\mathcal{L}_t}^{-1} \tag{9.3.2}$$

**Proposition 9.3.5.** *$h_t^T$ is a Liouville primitive on $\mathcal{L}_t$*

*Proof.* For the sake of simplicity, we omit $T$ and $t$. Let $(q,p) \in \mathcal{L}$ and $(q,\xi) = i_{S|\mathcal{L}}^{-1}(q,p)$. By definition of $S$, we have that $p = d_q S$ and $d_\xi S(q,\xi) = 0$.

$$\begin{aligned} dh(q,p) &= d(S \circ i_{S|\mathcal{L}}^{-1})(q,p) = dS(i_{S|\mathcal{L}}^{-1}(q,p)) \circ d i_{S|\mathcal{L}}^{-1}(q,p) \\ &= \begin{pmatrix} d_q S(q,\xi) & d_\xi S(q,\xi) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \times & \times \end{pmatrix} = \begin{pmatrix} p & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \times & \times \end{pmatrix} \\ &= p dq = \lambda(p,q) \end{aligned}$$

□

This gives us a path of Liouville primitives on $\mathcal{L}_t$. Now, we would like to construct a Liouville primitive on the Lagrangian submanifold $\mathscr{L}$.

Fix $h_0$ a Liouville primitive on $\mathcal{L}_0$. For all time $t \in \mathbb{R}$, define the map $h_t : \mathcal{L}_t \to \mathbb{R}$ by

$$h_t(\phi_H^t(x)) = h_0(x) + \int_0^t \left(\gamma_x^*\lambda - H(\tau, \gamma_x(\tau))\right) d\tau \tag{9.3.3}$$

where $x$ belongs to $\mathcal{L}$ and $\gamma_x(\tau) = \phi_H^\tau(x)$.

**Proposition 9.3.6.** *For all time $t \in \mathbb{R}$, $h_t$ is a Liouville primitive on $\mathcal{L}_t$.*

*Proof.* Fix a positive time $t > 0$. Let $x_t = \phi_H^t(x)$ and $y_t = \phi_H^t(y)$ be two points of $\mathcal{L}_t$ linked by a curve $\sigma_t = \phi_H^t(\sigma) : [0,1] \to \mathcal{L}_t$. Set $\gamma_x(\tau) = \phi_H^\tau(x)$ and its extended curve $\zeta_x(\tau) = \big(\tau, E_x(\tau), \gamma_x(\tau)\big) \in \mathscr{L}$. Since $\mathscr{L} \subset \{\mathscr{H} = 0\}$, for all $\tau$, $\mathscr{H}\big(\zeta_x(\tau)\big) = 0$ gives

$$E_x(\tau) = -H\big(\tau, \gamma_x(\tau)\big) \tag{9.3.4}$$

Similarly, define $\gamma_y(\tau) = \phi_H^\tau(y)$ and its extension $\zeta_y : [0,t] \to \mathscr{L}$. We denote by $\sigma : [0,1] \to \mathscr{L}$ the extention of $\sigma : [0,1] \to \mathcal{L}$. Now set $\overline{\zeta_x}(\tau) = \zeta_x(t-\tau)$ the time-backward curve corresponding to $\zeta_x$ defined on $\tau \in [0,t]$, and $\zeta = \overline{\zeta_x}.\sigma.\zeta_y$ the concatenation of the three curves. Then we have

$$\begin{aligned} h_t(y_t) - h_t(x_t) &= -\int_0^t \left(\gamma_x^*\lambda - H(\tau, \gamma_x(\tau))\right) d\tau + [h_0(y) - h_0(x)] + \int_0^t \left(\gamma_y^*\lambda - H(\tau, \gamma_y(\tau))\right) d\tau \\ &= \int_{\overline{\zeta_x}} \Lambda + \int_\sigma \Lambda + \int_{\zeta_y} \Lambda = \int_\zeta \Lambda = \int_{\phi_{\mathscr{H}}^{[0,t]}(\sigma)} d\Lambda + \int_{\sigma_t} \Lambda = 0 + \int_{\sigma_t} \Lambda = \int_{\sigma_t} \lambda \end{aligned}$$

where we used Stokes formula on the set $\phi_{\mathscr{H}}^{[0,t]}(\sigma) := \cup_{\tau \in [0,1]} \phi_{\mathscr{H}}^\tau(\sigma)$, and we used the fact that $\mathscr{L}$ is a Lagrangian submanifold of $(T^*\mathcal{M}, -d\Lambda)$. □

The wanted extended Liouville primitive is given by the map $\hbar : \mathscr{L} \to \mathbb{R}$ defined by

$$\hbar(t, E, q, p) = h_t(q, p) \tag{9.3.5}$$

where $E$ is such that $(t, E, q, p) \in \mathscr{L}$.

**Proposition 9.3.7.** *$\hbar$ is a Liouville primitive on $\mathscr{L}$.*

We omit the proof of this assertion as it is contained in the proof of the Proposition 9.3.6 above.

**Remark 9.3.8.** Note that for $s < t$ be two real times and $x \in \mathcal{L}_s$, if we set $\gamma(\tau) = \phi_H^{s,\tau}(x)$, then

$$h_t(\phi_H^{s,t}(x)) = h_s(x) + \int_s^t \left(\gamma^*\lambda - H(\tau, \gamma(\tau))\right) d\tau \tag{9.3.6}$$

In the following, we will link the primitive $\hbar$ to the constructed primitives $h_t^T$ and their generating functions $S_t^T$. Fix $T > |t|$, we know that $h_t$ and $h_t^T$ are two Liouville primitives for $\mathcal{L}_t$, hence they differ by a constant $\delta_t^T$ and $h_t = h_t^T + \delta_t^T$. This constant depends smoothly on $t$ since the paths $(h_t^T)_t$ and $(h_t)_t$ are smooth. Moreover, for the two generating functions $S_t^T$ and $S_t^T + \delta_t$, we have $i_{S_t^T} = i_{S_t^T + \delta_t}$.

Hence, by replacing $S_t^T$ by the g.f.q.i $S_t^T + \delta_t$, we can assume that for all $T > 0$ and $t \in [-T, T]$,

$$h_t = h_t^T = S_t^T \circ i_{S|\mathcal{L}_t}^{-1} \tag{9.3.7}$$

From now on, this will be assumed true.

**Proposition 9.3.9.** *Let $T > 0$ be a fixed real time. Then, with the notation $\mathscr{L}_{(-T,T)} = \mathscr{L} \cap \{\tau \in (-T,T)\}$ we have the equality*

$$\mathscr{L}_{(-T,T)} = \left\{ \left(\tau, \partial_\tau S^T(\tau,q,\xi), q, d_q S^T(\tau,q,\xi)\right) \mid d_\xi S^T(\tau,q,\xi) = 0,\ (\tau,q) \in (-T,T) \times M \right\} \tag{9.3.8}$$

In other words, the map $S^T$ plays the role of a g.f.q.i for $\mathscr{L}_{(-T,T)}$ in $T^*\mathcal{M} = T^*(\mathbb{R} \times M)$.

*Proof.* Let $\tau \in (-T,T)$ and let $(q,p)$ be a point of $\mathcal{L}_\tau$. By definition of the generating functions, we have that $p = d_q S^T(\tau,q,\xi)$ with $(q,\xi) = i_{S_\tau^T}^{-1}(q,p)$. Let $E \in \mathbb{R}$ be the energy such that $(\tau, E, q, p) \in \mathscr{L}$. We know from Theorem 9.3.2 that $S^T$ is regular with respect to $\tau$ and thus a derivation is possible. We need to prove that $E = \partial_\tau S^T(\tau,q,\xi)$.

We define the set $\Sigma_{S^T}$ by

$$\Sigma_{S^T} \coloneqq \left\{ (\tau,q,\xi) \in (-T,T) \times M \times \mathbb{R}^{k_T} \mid d_\xi S^T(\tau,q,\xi) = 0 \right\} \tag{9.3.9}$$

so that the map $S^T(\tau,q,\xi) = h_\tau \circ i_{S^T}(\tau,q,\xi)$ is defined on $\Sigma_{S^T}$. We need to differentiate with respect to $\tau$. To do so, we determine $T_{(\tau,q,\xi)}\Sigma_{S^T}$ given by differentiating the equation $d_\xi S^T(\tau,q,\xi) = 0$. Hence,

$$T_{(\tau,q,\xi)}\Sigma_{S^T} = \{ (\delta\tau, \delta q, \delta\xi) \mid d^2_{\tau\xi} S^T(\tau,q,\xi).\delta\tau + d^2_{q\xi} S^T(\tau,q,\xi).\delta q + d^2_{\xi\xi} S^T(\tau,q,\xi).\delta\xi = 0 \} \tag{9.3.10}$$

Moreover, since $S_\tau^T$ is a generating function of $\Sigma_{S_\tau^T}$, we know that the map $(\delta q, \delta\xi) \mapsto d^2_{q\xi} S^T(\tau,q,\xi).\delta q + d^2_{\xi\xi} S^T(\tau,q,\xi).\delta\xi$ is onto. Hence, for all $\delta\tau$, there exist $(\delta q, \delta\xi)$ such that $(\delta\tau, \delta q, \delta\xi) \in T_{(\tau,q,\xi)}\Sigma_{S^T}$.

Take such a vector $(\delta\tau, \delta q, \delta\xi) \in T_{(\tau,q,\xi)}\Sigma_{S^T}$. We have

$$\begin{aligned} dS^T(t,q,\xi).(\delta\tau,\delta q,\delta\xi) &= \partial_\tau S^T(\tau,q,\xi).\delta\tau + d_q S^T(\tau,q,\xi).\delta q + d_\xi S^T(\tau,q,\xi).\delta\xi \\ &= \partial_\tau S^T(\tau,q,\xi).\delta\tau + p\delta q \end{aligned}$$

and

$$\begin{aligned}
dS^T(t,q,\xi).(\delta\tau,\delta q,\delta\xi) &= d(h^T \circ i_{S^T})(t,q,\xi).(\delta\tau,\delta q,\delta\xi)\\
&= dh^T(t,q,p)\circ di_{S^T}(t,q,\xi).(\delta\tau,\delta q,\delta\xi)\\
&= \Lambda(t,q,p)\circ di_{S^T}(t,q,\xi).(\delta\tau,\delta q,\delta\xi)\\
&= \begin{pmatrix} E & p & 0\end{pmatrix}\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ d^2_{tq}S^T(t,q,\xi) & d^2_{qq}S^T(t,q,\xi) & 0\end{pmatrix}\begin{pmatrix}\delta\tau\\ \delta q\\ \delta\xi\end{pmatrix}\\
&= \begin{pmatrix} E & p & 0\end{pmatrix}\begin{pmatrix}\delta\tau\\ \delta q\\ \delta\xi\end{pmatrix} = E\delta\tau + p\delta q
\end{aligned}$$

where we use that $h$ derives from $\hbar$, Liouville primitive on $\mathscr{L}$. We obtain

$$(\partial_\tau S^T(\tau,q,\xi) - E)\delta\tau = p\delta q - p\delta q = 0$$

Taking $\delta\tau \neq 0$, which is possible by the study of $T_{(\tau,q,\xi)}\Sigma_{S^T}$, we conclude that $\partial_t S^T(t,q,\xi) = E$. Therefore, $(t,E,q,p)\in\mathscr{L}$ if and only if $p = d_q S^T(t,q,\xi)$ and $E = \partial_t S^T(t,q,\xi)$ with $(q,\xi) = i^{-1}_{S^T_t}(q,p)$. □

## 9.4 Spectral Invariants and Graph Selectors

In this section we present the construction and properties of spectral invariants rising from generating functions. Subsections 9.4.1 and 9.4.2 contain the construction and properties of these spectral invariants. And in Subsection 9.4.3 we introduce the notion of graph selectors which will play a fundamental role in the proof of the main result.

All material in this section is known and considered standard. Nevertheless, we chose to provide the proofs of all the properties needed in the final proof of this chapter, following following [Vit92]. This may provide sufficient background for the reader unfamiliar with spectral invariants and graph selectors.

### 9.4.1 Spectral Invariants

In this section, we define the spectral invariants of functions $S : E = \mathcal{M}\times\mathbb{R}^k \to \mathbb{R}$ that are quadratic at infinity, i.e such that there exist a real constant $c\in\mathbb{R}$ and a non-degenerate quadratic form $Q:\mathbb{R}^k\to\mathbb{R}$ such that $S = c+Q$ outside of a compact set of $E$. We will denote f.q.i for functions quadratic at infinity.

Fix such an f.q.i $S$ with associated pair $(c,Q)$. The domain $\mathbb{R}^k$ of the non-degenerate quadratic form $Q$ decomposes into $F^+ \oplus F^-$ which are the positive and negative spaces for $Q$ and $m = \dim F^-$ is the index of $Q$.

For any real number $a \in \mathbb{R}$, we denote the sub-level of $S$ with height $a$ by

$$S^a = \{e \in E \mid S(e) \leq a\} \tag{9.4.1}$$

Since $S = c + Q$ outside of a compact set, there exists a large $N > 0$ such that $S^N = \mathcal{M} \times Q^{N-c}$ and $S^{-N} = \mathcal{M} \times Q^{-N-c}$. Hence, for any $a \in \mathbb{R}$ and large $b \geq N$, the homotopy types of the pairs $(S^b, S^a)$ and $(S^a, S^{-b})$ are independent of $b$. We denote these respectively by $(S^\infty, S^a)$ and $(S^a, S^{-\infty})$.

Let $H^*$ denote the simplicial cohomology with coefficients in a field, namely $\mathbb{R}$. By a successive application of Künneth formula and Thom's isomorphism to the trivial fibre bundle $\mathcal{M} \times F^- \to \mathcal{M}$, one gets

$$H^*(S^\infty, S^{-\infty}) \simeq H^*(\mathcal{M}) \otimes H^*(Q^\infty, Q^{-\infty}) \simeq H^*(\mathcal{M}) \otimes H^*(\mathbb{D}^m, \partial\mathbb{D}^m) \simeq H^{*-m}(\mathcal{M}) \tag{9.4.2}$$

where $\mathbb{D}^m$ is the $m$-dimensional disk.

Moreover, the inclusion $S^a \hookrightarrow S^b$ for $a < b$ induces the cohomological morphism $H^*(S^b, S^{-\infty}) \hookrightarrow H^*(S^a, S^{-\infty})$. The existence of this map added to the fact that $H^*(S^{-\infty}, S^{-\infty}) = 0$ makes sense of the following definition.

**Definition 9.4.1.** Let $S : E = \mathcal{M} \times \mathbb{R}^k \to \mathbb{R}$ be a f.q.i. For all $\alpha \in H^*(\mathcal{M}) \smallsetminus \{0\}$ we define the spectral invariant $c(\alpha, S)$ by

$$c(\alpha, S) := \inf\{a \in \mathbb{R} \mid \alpha \neq 0 \text{ in } H^*(S^a, S^{-\infty})\} \tag{9.4.3}$$

**Remark 9.4.2.** For all real numbers $a < b < c$, the inclusions $S^a \hookrightarrow S^b \hookrightarrow S^c$ yield the commutative diagram

$$\begin{tikzcd} H^*(S^c, S^{-\infty}) \arrow[r] \arrow[dr] & H^*(S^b, S^{-\infty}) \arrow[d] \\ & H^*(S^a, S^{-\infty}) \end{tikzcd} \tag{9.4.4}$$

so that if $\alpha_q$ is null in $H^*(S^b, S^{-\infty})$, then it is null in $H^*(S^a, S^{-\infty})$ for all $a < b$, and we have

$$c(\alpha, S) = \sup\{a \in \mathbb{R} \mid \alpha = 0 \text{ in } H^*(S^a, S^{-\infty})\} \tag{9.4.5}$$

**Proposition 9.4.3.** *For any f.q.i $S$, $c(\alpha, S)$ is a critical value of $S$.*

*Proof.* Assume $c = c(\alpha, S)$ is not a critical value of $S$. Since $S$ is equal to a non-degenerate quadratic form outside of a compact set, its critical points must be contained in a compact set. Hence there exists a small $\varepsilon > 0$ such that $S$ has no critical values in $[c-\varepsilon, c+\varepsilon]$ and the gradient of $S$ is non-null in $S^{c+\varepsilon} \setminus \mathring{S}^{c-\varepsilon} = S^{-1}([c-\varepsilon, c+\varepsilon])$. Therefore, $S^{c+\varepsilon}$ can be contracted to $S^{c-\varepsilon}$ using a gradient flow generated by a vector field of the form $X = -\chi \frac{\nabla S}{\|\nabla S\|^2}$ where $\chi$ is a bump map identically equal to zero in $S^{c+\varepsilon} \setminus S^{c-\varepsilon}$. This shows that $H^*(S^{c+\varepsilon}, S^{c-\varepsilon}) = 0$. The long exact sequence for the triple $(S^{c+\varepsilon}, S^{c-\varepsilon}, S^{-\infty})$ is

$$H^*(S^{c+\varepsilon}, S^{c-\varepsilon}) \to H^*(S^{c-\varepsilon}, S^{-\infty}) \to H^*(S^{c+\varepsilon}, S^{-\infty}) \to H^{*+1}(S^{c+\varepsilon}, S^{c-\varepsilon}) \qquad (9.4.6)$$

with the two extremities being null. This results to the isomorphism

$$H^*(S^{c-\varepsilon}, S^{-\infty}) \simeq H^*(S^{c+\varepsilon}, S^{-\infty}) \qquad (9.4.7)$$

However, by the definition of $c$ we have $\alpha \neq 0$ in $H^*(S^{c+\varepsilon}, S^{-\infty})$ and $\alpha = 0$ in $H^*(S^{c-\varepsilon}, S^{-\infty})$. We get a contradiction to (9.4.7). □

As an application to generating functions, we obtain the following proposition.

**Proposition 9.4.4.** *Let $\mathcal{L}$ be an exact Lagrangian submanifold of $T^*\mathcal{M}$ with g.f.q.i $S$ and Liouville primitive $h = S \circ i^{-1}_{S|\mathcal{L}}$. Then for all $\alpha \in H^*(\mathcal{M})$,*

$$\min h \leq c(\alpha, S) \leq \max h \qquad (9.4.8)$$

*Proof.* We know from Proposition 9.4.3 that $c(\alpha, S)$ is a critical value of $S$. Then there exists $(q, \xi)$ such that $S(q, \xi) = c(\alpha, S)$ and $dS(q, \xi) = 0$. It follows that $d_\xi S(q, \xi) = 0$, $(q, \xi) \in \Sigma_S$ and

$$h(x) = S \circ i_S^{-1}(x) = S(q, \xi) = c(\alpha, S)$$

Therefore we conclude that $\min h \leq c(\alpha, S) \leq \max h$. □

For any f.q.i $S$, we denote by $\|S\|_\infty$ the quantity in $[0, +\infty]$ given by

$$\|S\|_\infty = \sup\{|S(q, \xi)| \mid (q, \xi) \in M \times \mathbb{R}^k\} \qquad (9.4.9)$$

**Proposition 9.4.5.** *For any f.q.i $S_1$, there exists a real number $\varepsilon_0 > 0$ small enough such that for all $0 < \varepsilon < \varepsilon_0$ and all f.q.i $S_2$ such that $\|S_2 - S_1\|_\infty \leq \epsilon$, we have $|c(\alpha, S_2) - c(\alpha, S_1)| \leq \varepsilon$.*

*Proof.* To simplify the notations, set $c_1 = c(\alpha, S_1)$. For $\varepsilon$ small enough, the non-degenerate quadratic forms at infinity associated to $S_1$ and $S_2$ have same index. Hence, we infer that $S_1^{\pm\infty}$ and $S_2^{\pm\infty}$ are homotopically equivalent and we obtain an isomorphism $H^*(S_1^\infty, S_1^{-\infty}) \simeq H^*(S_2^\infty, S_2^{-\infty})$ sending the element $\alpha_1$ corresponding to $\alpha$ in $H^*(S_1^\infty, S_1^{-\infty})$ to the element $\alpha_2$ corresponding to $\alpha$ in $H^*(S_2^\infty, S_2^{-\infty})$.

Now fix a real number $\delta > 0$. We know from the hypothesis that $S_2 \leq S_1 + \varepsilon$ and more precisely that $S_1^{c_1+\delta} \subset S_2^{c_1+\delta+\varepsilon}$. This inclusion yields the cohomological morphism

$$i^* : H^*(S_2^{c_1+\delta+\varepsilon}, S_2^{-\infty}) \to H^*(S_1^{c_1+\delta}, S_1^{-\infty}) \tag{9.4.10}$$

and we obtain a commutative diagram

$$\begin{array}{ccc} H^*(S_2^{\infty}, S_2^{-\infty}) & \xrightarrow{\sim} & H^*(S_1^{\infty}, S_1^{-\infty}) \\ \downarrow & & \downarrow \\ H^*(S_2^{c_1+\delta+\varepsilon}, S_2^{-\infty}) & \xrightarrow{i^*} & H^*(S_1^{c_1+\delta}, S_1^{-\infty}) \end{array} \tag{9.4.11}$$

By definition of $c_1$, we know that $i^*(\alpha_2) = \alpha_1 \neq 0$ in $H^*(S_1^{c_1+\delta}, S_1^{-\infty})$. Thus, $\alpha_2 \neq 0$ in $H^*(S_2^{c_1+\delta+\varepsilon}, S_2^{-\infty})$ meaning that $c(\alpha, S_2) \leq c_1 + \delta + \varepsilon$. By letting $\delta$ go to zero, we conclude that

$$c(\alpha, S_2) - c(\alpha, S_1) \leq \varepsilon \tag{9.4.12}$$

The symmetry between $S_1$ and $S_2$ yields the inverse inequality. □

### 9.4.2 Operations on Spectral Invariants

We focus in this subsection on two key operations on Lagrangian submanifolds and their induced properties on spectral invariants. These will show crucial in the proof of the main theorem.

**Definition 9.4.6.** 1. For a Lagrangian submanifold $\mathcal{L}$ in $T^*\mathcal{M}$ we define its inverted Lagrangian submanifold $\overline{\mathcal{L}}$ as

$$\overline{\mathcal{L}} := \{(q, -p) \mid (q,p) \in \mathcal{L}\} \tag{9.4.13}$$

If $S$ is a generating function of $\mathcal{L}$, $-S$ is a generating function of $\overline{\mathcal{L}}$.

2. For two Lagrangian submanifolds $\mathcal{L}_1$ and $\mathcal{L}_2$ in $T^*\mathcal{M}$ we define their fibred sum $\mathcal{L}_1 \# \mathcal{L}_2$ as the set

$$\mathcal{L}_1 \# \mathcal{L}_2 := \{(q, p_1 + p_2) \mid (q,p_1) \in \mathcal{L}_1,\ (q,p_2) \in \mathcal{L}_2\} \tag{9.4.14}$$

If $S_1(q, \xi_1)$ and $S_2(q, \xi_2)$ are two respective generating functions of $\mathcal{L}_1$ and $\mathcal{L}_2$, then we associate to $\mathcal{L}_1 \# \mathcal{L}_2$ the fibred sum $S_1 \oplus S_2$ given by

$$S_1 \oplus S_2(q, \xi_1, \xi_2) := S_1(q, \xi_1) + S_2(q, \xi_2) \tag{9.4.15}$$

**Remark 9.4.7.** Note that $\mathcal{L}_1 \# \mathcal{L}_2$ is not necessarily a submanifold and hence not a Lagrangian submanifold of $T^*\mathcal{M}$, but the fibred sum $S := S_1 \oplus S_2$ still generates $\mathcal{L}_1 \# \mathcal{L}_2$ in the sense that

$$\mathcal{L}_1\#\mathcal{L}_2 = \{(q, d_q S(q,\xi_1,\xi_2) \mid d_{(\xi_1,\xi_2)} S(q,\xi_1,\xi_2) = 0\}$$

The function $S_1 \oplus S_2$ is not quadratic at infinity. This issue is solved by the following proposition.

**Proposition 9.4.8.** *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two Lagrangian submanifolds generated by g.f.q.i $S_1$ and $S_2$ with corresponding constant and quadratic form pairs $(c_1, Q_1)$ and $(c_2, Q_2)$. Then, $\mathcal{L}_1\#\mathcal{L}_2$ is generated by a g.f.q.i $S$ with corresponding pair $(c_1+c_2, Q_1 \oplus Q_2)$, and there exists a diffeomorphism $\phi$ of $T^*\mathcal{M}$ such that $S\circ\phi = S_1 \oplus S_2$.*

*Proof.* For simplicity, we assume that $c_1 = c_2 = 0$. We set $S^0 := S_1 \oplus S_2$, $Q := Q_1 \oplus Q_2$. We have

$$S^0 - Q = S_1 \oplus S_2 - Q_1 \oplus Q_2 = (S_1 - Q_1) \oplus (S_2 - Q_2)$$

with the supports of $S_1 - Q_1$ and $S_2 - Q_2$ being both compact in their domains of definition. Hence, there exists a constant $C_1$ and $C_2 > 0$ such that

$$\|S_1 - Q_1\|_1 := \|S_1 - Q_1\|_\infty + \|\nabla(S_1 - Q_1)\|_\infty \leq C_1 \quad \text{and} \quad \|S_2 - Q_2\|_1 \leq C_2$$

and setting $C = C_1 + C_2$, we get

$$\|S^0 - Q\|_1 \leq \|S_1 - Q_1\|_1 + \|S_2 - Q_2\|_1 \leq C1 + C_2 = C \tag{9.4.16}$$

Let $B > A > 0$ be large real numbers and consider an increasing smooth map $\rho : [0, +\infty) \to [0,1]$ such that $\rho \equiv 0$ on $[0, A]$, $\rho \equiv 1$ on $[B + \infty)$ and $|\rho'| \leq \varepsilon$ small enough. We consider the map

$$S^1(q,\xi) = \rho(\xi)Q(q,\xi) + (1-\rho(\xi))S^0(q,\xi)$$

and the family $(S^t)_{t\in[0,1]}$ defined by

$$S^t = (1-t)S^0 + tS^1 = Q + (1 - t\rho)(S^0 - Q)$$

Note that $S^1$ is quadratic at infinity with quadratic form $Q$ on $\{|\xi| \geq B\}$.

We will construct an isotopy $\phi^t$ such that for all $t \in [0,1]$, $S^t \circ \phi^t = S^0$. We consider the vector flow $X_t = \frac{d\phi^t}{dt} \circ (\phi^t)^{-1}$ associated to $\phi^t$. We assume that $\phi^t$ preserves the fibres and we adopt the notations $\phi^t(q,\xi) = (q,\eta)$ and $X_t = (0, \sigma_t)$. We would like to get

$$\frac{d}{dt}(S^t \circ \phi^t) = 0 \tag{9.4.17}$$

Hence, computation yields

$$\begin{aligned}0 &= \frac{dS^t}{dt}(q,\eta) + dS^t.X_t(q,\eta)\\ &= \rho(\eta)(S^0 - Q)(q,\eta) + \langle \partial_\eta S^t(q,\eta), \sigma_t(q,\eta)\rangle\end{aligned}$$

with

$$\partial_\eta S^t(q,\eta) = \partial_\eta Q(q,\eta) - t\rho'(\eta)(S^0 - Q)(q,\eta) + (1 - t\rho)\partial_\eta(S^0 - Q)$$

We set $\sigma_t = 0$ for $|\eta| \le A$. For $|\eta| \ge A$, we verify that we can define $\sigma_t$ such that

$$\langle \partial_\eta S^t(q,\eta), \sigma_t(q,\eta)\rangle = -\rho(\eta)(S^0 - Q)(q,\eta) \tag{9.4.18}$$

Since the quadratic form $Q$ is non-degenerate, we deduce that there exists a constant $C' > 0$ such that $|\partial_\eta Q| \ge C'|\eta|$. And using (9.4.16), we get for $|\eta| \ge A$

$$\begin{aligned}|\partial_\eta S^t(q,\eta)| &\ge |\partial_\eta Q(q,\eta)| - \left|t\rho'(\eta)(S^0 - Q)(q,\eta)\right| - \left|(1 - t\rho)\partial_\eta(S^0 - Q)\right|\\ &\ge C'|\eta| - C(\varepsilon + 1) \ge C'A - C(\varepsilon + 1) \ge 2C'A\end{aligned}$$

for $A$ large enough. Therefore, the identity (9.4.18) allows to define $\sigma_t$ and we have

$$|\sigma_t(q,\eta)| \le \frac{|-\rho(\eta)(S^0 - Q)(q,\eta)|}{|\partial_\eta S^t(q,\eta)|} \le \frac{C}{2C'A}$$

which shows that the vector field $X_t$ is complete. The diffeomorphism $\phi := \phi^1$ and the g.f.q.i $S := S^1$ verify the desired properties of the statement. □

**Remark 9.4.9.** The proposition above implies that the level sets of $S_1 \oplus S_2$ and of the g.f.q.i $S$ are homotopic. Consequently we get the homotopy equivalence

$$\left((S_1 \oplus S_2)^{\infty}, (S_1 \oplus S_2)^{-\infty}\right) \simeq \left((Q_1 \oplus Q_2)^{\infty}, (Q_1 \oplus Q_2)^{-\infty}\right) \tag{9.4.19}$$

where $Q_1 \oplus Q_2$ is quadratic on fibres with index the sum of the indexes of $S_1$ and $S_2$.

This means that $S_1 \oplus S_2$ can be seen as a f.q.i and with well defined spectral invariants $c(\alpha, S_1 \oplus S_2)$ as in (9.4.3). We can that it plays the role of a g.f.q.i for the set $\mathcal{L}_1 \# \mathcal{L}_2$.

We now focus on the consequences of these two operations on the spectral invariants. We start by a property on the inversion operation.

**Proposition 9.4.10.** *Let* $1$ *and* $\mu$ *be the respective generators of* $H^0(\mathcal{M})$ *and* $H^d(\mathcal{M})$ *and let* $S : E = \mathcal{M} \times \mathbb{R}^k \to \mathbb{R}$ *be a q.f.i. Then, we have*

$$c(\mu, -S) = -c(1, S) \tag{9.4.20}$$

*Proof.* we first link the sublevels of the g.f.q.i $S$ to those of the g.f.q.i $-S$. Let $a$ be a real number.

$$(-S)^a = \{x \in E \mid -S(x) \leq a\} = E \setminus S^{-a} \tag{9.4.21}$$

Thus, by Alexander duality, we have the isomorphism

$$AD : H_m(S^{-a}, S^{-\infty}) \xrightarrow{\sim} H^{d+k-m}((-S)^a, (-S)^{-\infty}) \tag{9.4.22}$$

where $m$ is the index of $S$. And writing (vertically) the exact homological and cohomological sequences of the triplets $(S^{\infty}, S^{-a}, S^{-\infty})$ and $((-S)^{\infty}, (-S)^a, (-S)^{-\infty})$, we get the following commutative diagram

$$\begin{array}{ccc} H_m(S^{-a}, S^{-\infty}) & \xrightarrow{\sim} & H^{d+k-m}((-S)^{\infty}, (-S)^a) \\ \downarrow & & \downarrow \\ H_m(S^{\infty}, S^{-\infty}) & \xrightarrow{\sim} & H^{d+k-m}((-S)^{\infty}, (-S)^{-\infty}) \\ \downarrow & & \downarrow \\ H_m(S^{\infty}, S^{-a}) & \xrightarrow{\sim} & H^{d+k-m}((-S)^a, (-S)^{-\infty}) \end{array} \tag{9.4.23}$$

But before proceeding to a diagram chasing, we mention that the isomorphism (9.4.2) and its homological counterpart lead to

$$H^{d+k-m}((-S)^{\infty}, (-S)^{-\infty}) \simeq H^d(\mathcal{M}) \quad \text{and} \quad H_m(S^{\infty}, S^{-\infty}) \simeq H_0(\mathcal{M}) \tag{9.4.24}$$

Thus, we can see $\mu \in H^d(\mathcal{M})$ and its Poincaré dual $1_0 \in H_0(\mathcal{M})$ as respective elements of $H^{d+k-m}((-S)^{\infty}, (-S)^{-\infty})$ and $H_m(S^{\infty}, S^{-\infty})$. And with naturality $AD(1_0) = \mu$.

Suppose that $a < -c(1, S)$. Since $-a > c(1, S)$ and by definition of the spectral invariants, we know that $1 \neq 0$ in $H^m(S^{-a}, S^{-\infty})$. Then the morphism $H^m(S^{\infty}, S^{-\infty}) \to H^m(S^{-a}, S^{-\infty})$ is non null and so is its transpose map

$$H_m(S^{-a}, S^{-\infty}) \longrightarrow H_m(S^{\infty}, S^{-\infty}) \tag{9.4.25}$$

And since $\dim H_m(S^{\infty}, S^{-\infty}) = 1$, the observation above means that $1_0 \in H_m(S^{-a}, S^{-\infty})$ and $AD(1_0) \neq 0$ in $H^{d+k-m}((-S)^{\infty}, (-S)^a)$. By exactness of the vertical lines in the diagram, we get that $\mu = AD(1_0) = 0$ in $H^{d+k-m}((-S)^a, (-S)^{-\infty})$ and hence $a \leq c(\mu, -S)$. As a result,

$$-c(1, S) \leq c(\mu, -S) \tag{9.4.26}$$

Now, suppose that $a < c(\mu, -S)$. We know that $\mu = 0$ in $H^{d+k-m}((-S)^a, (-S)^{-\infty})$, then $\mu$ belongs to

$$\ker\left(H^{d+k-m}((-S)^{\infty}, (-S)^{-\infty}) \to H^{d+k-m}((-S)^a, (-S)^{-\infty})\right)$$

$$= \operatorname{Im}\left(H^{d+k-m}((-S)^\infty,(-S)^a) \to H^{d+k-m}((-S)^\infty,(-S)^{-\infty})\right) \quad (9.4.27)$$

And since $\mu \neq 0$ in $H^{d+k-m}((-S)^\infty,(-S)^{-\infty})$, the second morphism of (9.4.27) is non null and so is its left counterpart in the diagram (9.4.23). Hence, the transpose map

$$H^m(S^\infty, S^{-\infty}) \to H^m(S^{-a}, S^{-\infty}) \quad (9.4.28)$$

is non null. And since $\dim H^m(S^\infty, S^{-\infty}) = 1$, its generator 1 has a non null image in $H^m(S^{-a}, S^{-\infty})$. Therefore, $-a \geq c(1,S)$ and $a \leq -c(1,S)$. We obtain

$$c(\mu,-S) \leq -c(1,S) \quad (9.4.29)$$

The inequalities (9.4.26) and (9.4.29) give the desired property. □

**Proposition 9.4.11.** *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two exact Lagrangian submanifolds of $T^*\mathcal{M}$. Let $\phi^t$ be a Hamiltonian flow and let $S_{1,t}$ and $S_{2,t}$ be two (smooth) paths of generating functions respectively for $\phi^t(\mathcal{L}_1)$ and $\phi^t(\mathcal{L}_2)$ such that their associated Liouville primitives $h_{1,t}$ and $h_{2,t}$ do verify the identity (9.3.3). Then for all $\alpha \in H^*(\mathcal{M})$, the map $t \mapsto c(\alpha, S_{1,t} \ominus S_{2,t})$ is constant for $S_{1,t} \ominus S_{2,t} := S_{1,t} \oplus (-S_{2,t})$.*

*Proof.* From Proposition 9.4.3, we know that $c(\alpha, S_{1,t} \ominus S_{2,t})$ is a critical value of $S_{1,t} \ominus S_{2,t}$. Then there exists a critical point $(q_t,\xi_t,\eta_t) \in \mathcal{M} \times \mathbb{R}^{k_{S_1}} \times \mathbb{R}^{k_{S_2}}$ of $S_{1,t} \ominus S_{2,t}$ such that

$$c(\alpha, S_{1,t} \ominus S_{2,t}) = S_{1,t}(q_t,\xi_t) - S_{2,t}(q_t,\eta_t) \quad (9.4.30)$$

and

$$d_\xi S_{1,t}(q_t,\xi_t) = 0, \quad d_\eta S_{2,t}(q_t,\eta_t), \quad d_q S_{1,t}(q_t,\xi_t) = d_q S_{2,t}(q_t,\eta_t) \quad (9.4.31)$$

Set $x_t = \left(q_t, d_q S_{1,t}(q_t,\xi_t)\right) = \left(q_t, d_q S_{2,t}(q_t,\eta_t)\right) \in \phi^t(\mathcal{L}_1) \cap \phi^t(\mathcal{L}_2) = \phi^t(\mathcal{L}_1 \cap \mathcal{L}_2)$ and $x = (q,p) \in T^*\mathcal{M}$ such that $\gamma(t) := x_t = \phi^t(x)$. Then using (9.3.3), we get

$$\begin{aligned} c(\alpha, S_{1,t} \ominus S_{2,t}) &= h_{1,t}(x_t) - h_{2,t}(x_t) \\ &= \left[h_{1,0}(x) + \int_0^t \gamma^*\lambda - H(\tau,\gamma(\tau))\, dt\right] - \left[h_{2,0}(x) + \int_0^t \gamma^*\lambda - H(\tau,\gamma(\tau))\, dt\right] \\ &= h_{1,0}(x) - h_{2,0}(x) = S_1(q,\xi) - S_2(q,\eta) \end{aligned}$$

where $(q,\xi,\eta)$ is a critical point of $S_1 \ominus S_2$. Hence, $c(\alpha, S_{1,t} \ominus S_{2,t})$ is a critical values of $S_1 \ominus S_2$.

Suppose at first that $S_1 \ominus S_2$ has a finite number of critical values. Proposition 9.4.5 shows that the map $t \mapsto c(\alpha, S_{1,t} \ominus S_{2,t})$ is continuous. And since it takes its value in a discrete set, it must be constant.

For the general case, knowing that the critical values of $S_1 \ominus S_2$ are bounded, we can approximate $S_1 \ominus S_2$ by a g.f.q.i (up to homotopy in the sense of Proposition 9.4.8)

$S_1' \ominus S_2'$ generating $\mathcal{L}_1'\#\overline{\mathcal{L}_2'}$ and verifying the assumption above. We get that the map $t \mapsto c\left(\alpha, S_{1,t}' \ominus S_{2,t}'\right)$ is constant and by Proposition 9.4.5, it converges uniformly on compact sets to $t \mapsto c\left(\alpha, S_{1,t} \ominus S_{2,t}\right)$ as $S_1' \ominus S_2'$ nears $S_1 \ominus S_2$. Therefore, the latter map is also constant. □

### 9.4.3 Graph Selectors

We recall in this subsection the standard graph selector principal [Cha91, PPS03, Ott15]. As in these references, our approach is done in the g.f.q.i framework. However, an interested reader may refer to [Oh97] and [AOdS18] for a more general approach based on Floer Homology.

#### Generalities on Graph Selectors

**Definition 9.4.12.** Let $\mathcal{L}$ be an exact Lagrangian submanifold of $T^*\mathcal{M}$ generated by a g.f.q.i $S : \mathcal{M} \times \mathbb{R}^k \to \mathbb{R}$. For all $q \in \mathcal{M}$, denote by $\alpha_q$ the generator of $H^*(\{q\})$. The *graph selector* of $\mathcal{L}$ associated to $S$ is the map $u_S : \mathcal{M} \to \mathbb{R}$ defined by

$$u_S(q) = c(\alpha_q, S_q) \quad \text{for } S_q := S_{|\{q\}\times\mathbb{R}^k} \tag{9.4.32}$$

**Proposition 9.4.13.** *The graph selector $u_S$ defined as above verify the following properties.*

1. *$u_S$ is Lipschitz on $\mathcal{M}$.*
2. *There exists an open subset $U \subset \mathcal{M}$ of full Lebesgue measure such that $u_S$ is as regular as $S$ on $U$ and for all $q \in U$,*

$$(q, d_q u_S) \in \mathcal{L} \quad \textit{and} \quad u_S(q) = h(q, d_q u_S) \tag{9.4.33}$$

*with $h = S \circ i_S^{-1}$ the Liouville primitive on $\mathcal{L}$ relative to the g.f.q.i $S$.*

*Proof.* 1. Let $d$ be a Riemannian metric on $\mathcal{M}$. Since the map $(q, q', \xi) \in \mathcal{M}^2 \times \mathbb{R}^k \mapsto S(q,\xi) - S(q',\xi) \in \mathbb{R}$ is of compact support, it is Lipschitz and there exists a constant $K > 0$ such that

$$\|S_{|\{q\}\times\mathbb{R}^k} - S_{|\{q'\}\times\mathbb{R}^k}\|_\infty \le K.d(q,q')$$

Then, one can adapt the proof of Proposition 9.4.5 and conclude that $u_S$ is $K$-Lipschitz.

2. Let $U_0$ be the open set of regular values of the $C^1$ projection $\pi_S : \Sigma_S \to \mathcal{M}$. As $\dim \Sigma_S = \dim \mathcal{M}$, Sard's theorem can be applied to the projection map $\pi_S$ and we deduce that $U_0$ is of full measure. Let $q$ be a point of $U_0$. $\Sigma_S$ is transverse to $\{q\}\times\mathbb{R}^k$,

then there exists a connected neighbourhood $V_q$ of $q$ in $\mathcal{M}$ such that $\pi_S^{-1}(V_q)$ is the disjoint union of $i$ graphs of $\xi_j : V_q \to \mathbb{R}^k$. Thus, $\mathcal{L}$ is the disjoint union of $i$ graphs of $y \in V_z \mapsto (y, d_qS(y, \xi_j(y))) \in T^*\mathcal{M}$ over $V_q$. Observe from the disjoint union that $d_qS(y, \xi_j(y))$ are pairwise distinct, then the sets $\{y \in V_q \mid S(y, \xi_{j_1}(y)) = S(y, \xi_{j_2}(y))\}$ are discrete.

Now set $U \subset U_0$ be the set of $q$ in $U_0$ such that the critical points of $S_q$ have pairwise distinct critical values. The observation above shows that $U$ is a full measure open set of $\mathcal{M}$. Fix a point $q$ in $U$ and let $V_q$ be its corresponding neighbourhood defined as above. By Proposition 9.4.3, for all $y \in V_q$, $u_S(y)$ is one of the critical values of $S_y$ and more precisely one of the pairwise distinct values $S(y, \xi_j(y))$. And by continuity of the maps $y \mapsto u_S(y)$ and $y \mapsto S(y, \xi_j(y))$ and by connectedness of $V_q$, there exists a unique $j_q$ such that for all $y$ in $V_q$, $u_S(y) = S(y, \xi_{j_x}(y))$. Therefore, $u_S$ is as regular as $S$ at $q$ and $du_S(q) = d_qS(q, \xi_{j_q}(q))$. This equality yields $i_S(q, \xi_{j_q}(q)) = (q, d_q u_S) \in \mathcal{L}$ and

$$h(q, d_q u_S) = S \circ i_S^{-1}(q, d_q u_S) = S(q, \xi_{j_q}(q)) = u_S(q)$$

□

**Proposition 9.4.14.** *Let $\mathcal{L}$ be an exact Lagrangian submanifold of $T^*\mathcal{M}$ generated by a g.f.q.i $S : \mathcal{M} \times \mathbb{R}^k \to \mathbb{R}$. We have*

$$u_{-S} = -u_S \tag{9.4.34}$$

*Proof.* The proof of $c(\alpha_q, -S_{|\{q\}\times\mathbb{R}^k}) = -c(\alpha_q, S_{|\{q\}\times\mathbb{R}^k})$ is a simpler counterpart of the proof of Proposition 9.4.10 restricted to the fibres. □

**Proposition 9.4.15.** *Let $\mathcal{L}$ be an exact Lagrangian submanifold of $T^*\mathcal{M}$ generated by a g.f.q.i $S : \mathcal{M} \times \mathbb{R}^k \to \mathbb{R}$. Let 1 and $\mu$ be the respective generators of $H^0(\mathcal{M})$ and $H^d(\mathcal{M})$. Then we have the bounds*

$$c(1, S) \le u_S \le c(\mu, S) \tag{9.4.35}$$

*Proof.* For $q \in \mathcal{M}$, set $S_q := S_{|\{q\}\times\mathbb{R}^k}$. Let $a$ be a real number. From the isomorphism depicted in (9.4.2), we have the two identifications $H^*(\mathcal{M}) \simeq H^*(S^\infty, S^{-\infty})$ and $H^*(\{q\}) \simeq H^*(S_q^\infty, S_q^{-\infty})$. Additionally, the inclusion $i_q : \{q\} \hookrightarrow \mathcal{M}$ yields the morphisms $i_q^* : H^*(S^\infty, S^{-\infty}) \to H^*(S_q^\infty, S_q^{-\infty})$ and $i_{q,a}^* : H^*(S^a, S^{-\infty}) \to H^*(S_q^a, S_q^{-\infty})$ that send 1 to $\alpha_q$.
Thus, the inclusions $S_q^a \hookrightarrow S^a$ and $S^a \hookrightarrow S^\infty$ complete the following commutative diagram

$$\begin{tikzcd}
H^*(\mathcal{M}) \simeq H^*(S^\infty, S^{-\infty}) \arrow[r, "i_q^*"] \arrow[d] & H^*(\{q\}) \simeq H^*(S_q^\infty, S_q^{-\infty}) \arrow[d] \\
H^*(S^a, S^{-\infty}) \arrow[r, "i_{q,a}^*"] & H^*(S_q^a, S_q^{-\infty})
\end{tikzcd} \tag{9.4.36}$$

A diagram chasing provides the first inequality. If $a > c(\alpha_q, S_{|\{q\}\times\mathbb{R}^k})$, by definition of the spectral invariants $i^*_{a,q}(1) = \alpha_q \neq 0$ in $H^*(S_q^a, S_q^{-\infty})$. Hence, $1 \neq 0$ in $H^*(S^a, S^{-\infty})$ and we get the inequality $u_S(q) = c(\alpha_q, S_q) \geq c(1, S)$.

We prove the second inequality using the Propositions 9.4.10 and 9.4.14. Applying the first part of the proof to $\overline{\mathcal{L}}$, we get

$$c(\alpha_q, -S_q) \geq c(1, -S) = -c(\mu, S)$$

and

$$u_S(q) = -u_{-S}(q) = -c(\alpha_q, -S_q) \leq c(\mu, S)$$

□

Recall that the *oscillation* of a scalar map $f : X \to \mathbb{R}$ defined on a compact set $X$ is the quantity

$$Osc(f) = \max f - \min f \tag{9.4.37}$$

**Corollary 9.4.16.** *With the notations of Proposition 9.4.15, if we consider $h$ the Liouville primitive on $\mathcal{L}$ relative to $S$, then we have*

$$\min h \leq u_S \leq \max h \tag{9.4.38}$$

*and in particular*

$$Osc(u_S) \leq Osc(h) \tag{9.4.39}$$

Note that this inequality is valid for all Liouville primitives on $\mathcal{L}$.

*Proof.* This is an immediate consequence of Propositions 9.4.15 and 9.4.4. □

**Proposition 9.4.17.** *Let $S_1$ and $S_2$ be two g.f.q.i and consider $S = S_1 \oplus S_2$. Fix a point $q$ in $\mathcal{M}$ and let $\alpha_{1,q}$ and $\alpha_{2,q}$ be the generators of $H^*(S_{1,q}^\infty, S_{1,q}^{-\infty})$ and $H^*(S_{2,q}^\infty, S_{2,q}^{-\infty})$. Then $\alpha_q = \alpha_{1,q} \otimes \alpha_{2,q}$ generates $H^*(S^\infty, S^{-\infty})$ and we have the inequality*

$$c(\alpha_{1,q}, S_{1,q}) + c(\alpha_{2,q}, S_{2,q}) = c(\alpha_{1,q} \otimes \alpha_{2,q}, S_{1,q} \oplus S_{2,q}) = c(\alpha_q, S_q) \tag{9.4.40}$$

*Proof.* Let $m_1$, $m_2$ and $m = m_1 + m_2$ be the respective indices of $S_{1,q}$, $S_{2,q}$ and $S_q$. We have $H^*(S_{1,q}^\infty, S_{1,q}^{-\infty}) \simeq H^*(\mathbb{D}^{m_1}, \partial\mathbb{D}^{m_1})$, $H^*(S_{2,q}^\infty, S_{2,q}^{-\infty}) \simeq H^*(\mathbb{D}^{m_2}, \partial\mathbb{D}^{m_2})$ and $H^*(S_q^\infty, S_q^{-\infty}) \simeq H^*(\mathbb{D}^m, \partial\mathbb{D}^m)$ which yields the isomorphism

$$\begin{aligned} H^*(S_q^\infty, S_q^{-\infty}) &\xrightarrow{\sim} H^*(S_{1,q}^\infty, S_{1,q}^{-\infty}) \otimes H^*(S_{2,q}^\infty, S_{2,q}^{-\infty}) \\ \alpha_q &\longmapsto \alpha_{1,q} \otimes \alpha_{2,q} \end{aligned}$$

Fix two real numbers $a$ and $b$. Then, we have the inclusions $S^a_{1,q} \times S^b_{2,q} \hookrightarrow S^{a+b}_q$ and for large $N$ and larger $N' > 0$, we have inclusion $(S^a_{1,q} \times S^{-N'}_{2,q}) \cup (S^{-N'}_{1,q} \times S^b_{2,q}) \hookrightarrow S^{-N}_q$. These and the Künneth formila yield the morphism

$$H^*(S^{a+b}_q, S^{-\infty}_q) \longrightarrow H^*\left(S^a_{1,q} \times S^b_{2,q}, (S^a_{1,q} \times S^{-\infty}_{2,q}) \cup (S^{-\infty}_{1,q} \times S^b_{2,q})\right) = H^*(S^a_{1,q}, S^{-\infty}_{1,q}) \otimes H^*(S^b_{2,q}, S^{-\infty}_{2,q})$$

Hence, the inclusions $S^a_{1,q} \hookrightarrow S^\infty_{1,q}$, $S^a_{2,q} \hookrightarrow S^\infty_{2,q}$ and $S^{a+b}_q \hookrightarrow S^\infty_q$ complete the following commutative diagram

$$\begin{array}{ccc} H^*(S^\infty_q, S^{-\infty}_q) & \xrightarrow{\sim} & H^*(S^\infty_{1,q}, S^{-\infty}_{1,q}) \otimes H^*(S^\infty_{2,q}, S^{-\infty}_{2,q}) \\ \downarrow & & \downarrow \\ H^*(S^{a+b}_q, S^{-\infty}_q) & \longrightarrow & H^*(S^a_{1,q}, S^{-\infty}_{1,q}) \otimes H^*(S^b_{2,q}, S^{-\infty}_{2,q}) \end{array} \tag{9.4.41}$$

Let $a > c(\alpha_{1,q}, S_{1,q})$ and $b > c(\alpha_{2,q}, S_{2,q})$. We have $\alpha_{1,q} \neq 0$ and $\alpha_{2,q} \neq 0$ respectively in $H^*(S^\infty_{1,q}, S^{-\infty}_{1,q})$ and $H^*(S^\infty_{2,q}, S^{-\infty}_{2,q})$. Then by the commutativity of the diagram, the lower arrow sends $\alpha_q$ to $\alpha_{1,q} \otimes \alpha_{2,q} \neq 0$ and we deduce that $\alpha_q \neq 0$ in $H^*(S^{a+b}_q, S^{-\infty}_q)$. Hence, $a + b \geq c(\alpha_q, S_q)$ and we deduce the inequality

$$c(\alpha_{1,q}, S_{1,q}) + c(\alpha_{2,q}, S_{2,q}) \geq c(\alpha_q, S_q)$$

Now, let $c < c(\alpha_{1,q}, S_{1,q}) + c(\alpha_{2,q}, S_{2,q})$. If $c = a + b$, then $a < c(\alpha_{1,q}, S_{1,q})$ or $b < c(\alpha_{2,q}, S_{2,q})$ and in particular $\alpha_{1,q} \otimes \alpha_{2,q}$ vanishes on $S^a_{1,q} \times S^b_{2,q}$. Hence, $\alpha_q = \alpha_{1,q} \otimes \alpha_{2,q}$ vanishes on $S^c_q = \bigcup_{t\in\mathbb{R}} S^{c-t}_{1,q} \times S^t_{2,q}$ and we deduce that $c \leq c(\alpha_q, S_q)$. We obtain the inverse inequality

$$c(\alpha_{1,q}, S_{1,q}) + c(\alpha_{2,q}, S_{2,q}) \leq c(\alpha_q, S_q)$$

□

**Proposition 9.4.18.** *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two Lagrangian submanifolds of $T^*\mathcal{M}$ respectively generated by the g.f.q.i $S_1$ and $S_2$. Let $u_{S_1}$ and $u_{S_2}$ be the corresponding graphs selectors. And let $1$ and $\mu$ be the respective generators of $H^0(\mathcal{M})$ and $H^d(\mathcal{M})$. Then we have the following bounds*

$$c(1, S_1 \ominus S_2) \leq u_{S_1} - u_{S_2} \leq c(\mu, S_1 \ominus S_2) \tag{9.4.42}$$

*Proof.* For $S := S_1 \ominus S_2$, an application of Propositions 9.4.14 and 9.4.17 yields

$$u_{S_1}(q) - u_{S_2}(q) = u_{S_1}(q) + u_{-S_2}(q) = c(\alpha_q, S_q)$$

and we deduce from Proposition 9.4.15 that

$$c(1,S) \leq u_{S_1}(q) - u_{S_2}(q) \leq c(\mu,S)$$

□

### Construction of the Graph Selector of $\mathscr{L}$

Let us summarize the constructions till this point of the chapter. We constructed $\hbar$ a Liouville primitive on the extended non compact exact Lagrangian submanifold $\mathscr{L}$ defined in Section 9.2. We had a path $(h_t)_{t\in\mathbb{R}}$ of Liouville primitives on $\mathcal{L}_t$ with the property that for all $t\in\mathbb{R}$ and $(q,p)\in\mathcal{L}_t$,

$$\hbar(t,E,q,p) = h_t(q,p)$$

where $E$ is such that $(t,E,q,p)\in\mathscr{L}$. And for all times $T>0$, we set the paths $(S_t^T : M\times\mathbb{R}^{k_T}\to\mathbb{R})_{t\in[-T,T]}$ of g.f.q.i such that $S_t^T$ generates $\mathcal{L}_t = \phi_H^t(\mathcal{L})$ and

$$h_t = S_t^T \circ i_{S|\mathcal{L}_t}^{-1}$$

Now, for all $T>0$ we define the graph selector of $\mathcal{L}_t$ as

$$u^T(t,q) = c(\alpha_q, S^T_{t|\{q\}\times\mathbb{R}^k}) \tag{9.4.43}$$

**Proposition 9.4.19.** *For all $t\in[-T,T]$, $u_t^T(q) \coloneqq u^T(t,\cdot)$ is independent of $T$.*

*Proof.* Fix $0<T<T'$ and $t\in[-T,T]$. The g.f.q.i $S_t^T$ and $S_t^{T'}$ generate $\mathcal{L}_t$, then by the uniqueness Theorem 9.3.4 they are equivalent. These two generating functions have been constructed according to (9.3.7) that is $S_t^T\circ i_{S_t^T}^{-1} = S_t^{T'}\circ i_{S_t^{T'}}^{-1}$. This means that the translation never occurs in the successive operations intervening in the equivalence between $S_t^T$ and $S_t^{T'}$. Thus, we only need to take care of isomorphism and stabilization operations.

We first deal with the isomorphism operation. Suppose that there exists a bundle isomorphism $\psi : M\times\mathbb{R}^{k_{T'}}\to M\times\mathbb{R}^{k_T}$ such that $S_t^{T'} = S_t^T\circ\psi$. Thus, for all $a\in\mathbb{R}$ and $q\in M$, $(S_t^{T'})_q^a \simeq (S_t^T)_q^a$ and consequently $c(\alpha_q,(S_t^{T'})_q) = c(\alpha_q,(S_t^T)_q)$ i.e $u_t^{T'}(q) = u_t^T(q)$.

We deal now with the stabilization operation. Suppose that $k_{T'} = k_T + k$ and that there exists a non-degenerate quadratic form $Q : M\times\mathbb{R}^k\to\mathbb{R}$ such that $S_t^{T'} = S_t^T\oplus Q$. The quadratic form $Q_q = Q_{|q\times\mathbb{R}^k}$ is non-degenerate so that its only critical value in the fibre of $q$ is $c(\alpha_q^Q, Q_q) = 0$. Hence, we deduce from Proposition 9.4.17 that $u_t^{T'}(q) = u_t^T(q) + c(\alpha_q^Q,Q_q) = u_t^T(q)$. □

This independence enables us to define the map $u = u_{\mathscr{L}} : \mathcal{M} = \mathbb{R}\times M\to\mathbb{R}$ given by

$$u_{\mathscr{L}}(t,q) = u_t^T(q) \quad \text{for some } T > |t| \tag{9.4.44}$$

**Proposition 9.4.20.** *$u_{\mathscr{L}}$ is a graph selector for the exact Lagrangian submanifold $\mathscr{L}$ in the sense that it verifies the properties of Proposition 9.4.13. More precisely*

1. *$u_{\mathscr{L}}$ is locally Lipschitz.*

2. *There exists an open subset $\mathcal{U} \subset \mathcal{M}$ of full Lebesgue measure such that $u_{\mathscr{L}}$ is as regular as $\hbar$ on $\mathcal{U}$ and for all $(t,q) \in \mathcal{U}$,*

$$\big(t, \partial_t u(t,q), q, d_q u(t,q)\big) \in \mathscr{L} \quad \textit{and} \quad u(t,q) = \hbar\big(t, \partial_t u(t,q), q, d_q u(t,q)\big) \tag{9.4.45}$$

*with $\hbar$ the Liouville primitive on $\mathscr{L}$ defined in Subsection 9.3.2.*

*Proof.* 1. Let $d$ be a Riemannian metric on $\mathcal{M} = \mathbb{R} \times M$. Let $(t,q)$ be a point of $\mathcal{M}$ and $T > |t|$ be a fixed time. Since the map $(t,q,t',q',\xi) \in \big([-T,T] \times M\big)^2 \times \mathbb{R}^k \mapsto S(t,q,\xi) - S(t',q',\xi) \in \mathbb{R}$ is of compact support, it is Lipschitz and there exists a constant $K_T > 0$ such that

$$\|S_{|\{(t,q)\}\times\mathbb{R}^k} - S_{|\{(t',q')\}\times\mathbb{R}^k}\|_\infty \leq K_T.d\big((t,q),(t',q')\big)$$

Then, one can apply Proposition 9.4.5 and conclude that $u_{\mathscr{L}}$ is $K_T$-Lipschitz on $[-T,T] \times M$. This being true for all $T > 0$, we conclude that $u_{\mathscr{L}}$ is locally Lipschitz on $\mathcal{M}$.

2. Observe that for $T > 0$ and $(t,q) \in (-T,T)\times M$, we have the equality $c(\alpha_q, S^T_{t|\{q\}\times\mathbb{R}^k}) = c(\alpha_{(t,q)}, S^T_{|\{(t,q)\}\times\mathbb{R}^k})$. Then, the proof of the second statement of Proposition 9.4.13 applied to $S^T$ instead of $S^T_t$ gives an open set $\mathcal{U}^T \subset (-T,T) \times M$ of full measure such that every element $(t,q)$ of $\mathcal{U}^T$ has a neighbourhood $\mathcal{V}_{(t,q)}$ and a regular map $\xi : \mathcal{V}_{(t,q)} \to \mathbb{R}^k$ such that for all $(s,y) \in \mathcal{V}_{(t,q)}$, $u_{\mathscr{L}}(s,y) = S^T\big(s,y,\xi(s,y)\big)$ which is as regular as $\hbar$. When differentiated, this gives

$$\partial_t u_{\mathscr{L}}(s,y) = \partial_t S^T\big(s,y,\xi(s,y)\big) \quad \text{and} \quad d_q u_{\mathscr{L}}(s,y) = d_q S^T\big(s,y,\xi(s,y)\big) \tag{9.4.46}$$

Therefore, Proposition 9.3.9 imply that $\big(t, \partial_t u(t,q), q, d_q u(t,q)\big)$ belongs to $\mathscr{L}$ meaning that

$$u(t,q) = u^T_t(q) = h_t(q, d_q u^T(q)) = \hbar\big(t, \partial_t u(t,q), q, d_q u(t,q)\big)$$

Finally, the set $\mathcal{U} = \bigcup_{T\in\mathbb{N}^*} \mathcal{U}^T$ of $\mathcal{M}$ is open of full measure and meets the required properties.

∎

## 9.5 The Birkhoff Theorem for the Lagrangian Submanifold $\mathscr{L}$

The current section is devoted to the proof of Theorem 9.1.5. We start in Subsection 9.5.1 by presenting some concepts that emerged in Fathi's weak-KAM theory that will show crucial in showing the main result. See [Fat08] [CI99] for elaborate expositions on the subject in the autonomous framework. The remainder of the section is full focused on the proof.

### 9.5.1 Calibration

The Hamiltonian $H : \mathbb{T}^1 \times T^*M \to \mathbb{R}$ is assumed to be Tonelli. This enables to define its convex conjugate named the *Lagrangian* $L : \mathbb{T}^1 \times TM \to \mathbb{R}$ given by the following formula

$$L(t, q, v) = \max_{p \in T_q^* M} \{p(v) - H(t, q, p)\} \tag{9.5.1}$$

We introduce a tool derived from convex analysis that will help in the upcoming proofs.

**Lemma 9.5.1.** *(Fenchel's inequality) For all $q$ in $M$ and all $(v, p) \in T_qM \times T_q^*M$*

$$p(v) \leq H(t, q, p) + L(t, q, v) \tag{9.5.2}$$

*with equality if and only if $p = \partial_v L(q, v)$ if and only if $v = \partial_p H(t, q, p)$.*

**Remark 9.5.2.** Note that for $x = (q, p)$, if $\phi_H^t(x) = (q(t), p(t))$ is a curve that follows the Hamiltonian flow, then the Hamiltonian equations results in the equalities

$$\dot{q}(t) = \partial_p H\big(t, q(t), p(t)\big) \quad \text{and} \quad p(t) = \partial_v L\big(t, q(t), \dot{q}(t)\big) \tag{9.5.3}$$

Another fundamental concept that has emerged from weak-KAM theory is the following.

**Definition 9.5.3.** Let $\gamma : [a, b] \to M$ be a $C^1$ curve on $M$. The *defect of calibration* of $\gamma$ is defined as

$$\delta(u, \gamma) = \int_a^b L(s, \gamma(s), \dot{\gamma}(s))\, ds - [u(b, \gamma(b)) - u(a, \gamma(a))] \tag{9.5.4}$$

Recall that $u$ is the graph selector of the extended Lagrangian submanifold $\mathscr{L}$ constructed in Subsection 9.4.3.

**Proposition 9.5.4.** *The defect of calibration $\delta$ is always non-negative. We say that the map $u : \mathbb{R} \times M \to \mathbb{R}$ is dominated by $L$.*

*Proof.* Let $\gamma : [a,b] \to M$ be a $C^1$ curve on $M$ such that for almost every time $t \in [a,b]$, $(t,\gamma(t))$ belongs to the dense open set $\mathcal{U}$ defined in Proposition 9.4.20.

$$\begin{aligned} u(b,\gamma(b)) - u(a,\gamma(a)) &= \int_a^b du(s,\gamma(s)).(1,\dot\gamma(s))\, ds = \int_a^b \partial_t u(s,\gamma(s)) + d_q u(s,\gamma(s)).\dot\gamma(s)\, ds \\ &\le \int_a^b \partial_t u(s,\gamma(s)) + H\big(s,\gamma(s),d_q u(s,\gamma(s))\big) + L\big(s,\gamma(s),\dot\gamma(s)\big)\, ds \\ &= \int_a^b \mathscr{H}\big(s,\partial_t u(s,\gamma(s)),\gamma(s),d_q u(s,\gamma(s))\big)\, ds + \int_a^b L(s,\gamma(s),\dot\gamma(s))\, ds \end{aligned}$$

where in the second line we used the Fenchel inequality (9.5.2).

Now by Proposition 9.4.20, we have from the assumption on $\gamma$ that for almost all $s \in [a,b]$,

$$\big(s,\partial_t u(s,\gamma(s)),\gamma(s),d_q u(s,\gamma(s))\big) \in \mathscr{L} \subset \{\mathscr{H} = 0\} \tag{9.5.5}$$

Then we get

$$\int_a^b \mathscr{H}\big(s,\partial_t u(s,\gamma(s)),\gamma(s),d_q u(s,\gamma(s))\big)\, ds = 0$$

and

$$\delta(u,\gamma) = \int_a^b L(s,\gamma(s),\dot\gamma(s))\, ds - [u(b,\gamma(b)) - u(a,\gamma(a))] \ge 0$$

The general configuration case is given by the perturbation process described in Lemma 9.5.5. ☐

**Lemma 9.5.5.** *Let $\gamma : [a,b] \to M$ be a $C^1$ curve and let $\mathcal{U}$ be a full Lebesgue measure set of $\mathcal{M} = \mathbb{R} \times M$. Then there exists a sequence of $C^1$ curves $\gamma_k : [a,b] \to M$ such that*

*i. For almost all $t \in [a,b]$, $(t,\gamma_k(t)) \in \mathcal{U}$.*

*ii. The sequence $(\gamma_k)_k$ converges to $\gamma$ in the $C^1$-topology.*

*Proof.* $\gamma : [a,b] \to M$ is a $C^1$ curve in $M$, then we can extend it to a $C^1$ curve defined on $(c,d) \supset [a,b]$. The curve $(t,\gamma(t))$ is $C^1$ embedded in $\mathcal{M} = \mathbb{R} \times M$, then it admits a tubular neighbourhood $\mathcal{O}$ and there exists a $C^1$ embedding $\psi : \mathcal{O} \to \mathbb{R} \times \mathbb{R}^n$ such that for all $t \in (c,d)$, $\psi(t,\gamma(t)) = (t,0)$.

Take $\mathcal{R}' = [a,b] \times [-\varepsilon,\varepsilon]^n \subset \psi(\mathcal{O})$ and $\mathcal{R} = \psi^{-1}(\mathcal{R}')$. Denote by $l$ the Lebesgue measure on $\mathcal{M}$ and $l' = \psi_*(l)$ be a measure on $\psi(\mathcal{O})$. Since $\mathcal{U}_0 := \mathcal{U} \cap \mathcal{R}$ is of full measure in $\mathcal{R}$, $\mathcal{U}_0' := \psi(\mathcal{U}_0)$ is of full measure in $\mathcal{R}'$. Then using Fubini

$$l'(\mathcal{R}') = l'(\mathcal{U}_0') = \int_{[-\varepsilon,\varepsilon]^n} \int_{([a,b]\times\{q'\})\cap\mathcal{U}_0'} dt'\, dq' \le \int_{[-\varepsilon,\varepsilon]^n} \int_{([a,b]\times\{q'\})} dt'\, dq' = l'(\mathcal{R}')$$

Hence, by positivity of the integrands, we get that for almost every $q'$ in $[-\varepsilon,\varepsilon]^n$,

$$\int_{([a,b]\times\{q'\})\cap\mathcal{U}_0'} dt' = \int_{([a,b]\times\{q'\})} dt' \tag{9.5.6}$$

or in other words, for all such $q'$, for almost every $t'$ in $[a,b]$, $(t',q')$ belongs to $\mathcal{U}_0$.

Let $(q'_k)_k$ be a sequence in $[-\varepsilon,\varepsilon]^n$ such that every element $q_k$ verifies the equality (9.5.6) and the sequence converges to 0 as $k$ goes to infinity. For all $k$, we define the constant curves $\gamma'_k \equiv q'_k : [a,b] \to \mathcal{R}'$ and $\gamma_k(t) = \psi^{-1}(t,\gamma'_k)$. The curves $\gamma'_k$ converge to $\psi(\gamma)$ in the $C^1$-topology and since $\psi$ is regular, the same holds for the curves $\gamma_k$. Moreover, from (9.5.6) we know that for almost all $t$ in $[a,b]$, $(t,\gamma_k(t))$ belongs to $\mathcal{U}$ which concludes the proof. □

Now that the domination of $u$ has been established, we can take more interest in curves that achieve equality in this domination, or in other words curves having null calibration defect.

**Definition 9.5.6.** A $C^1$ curve $\gamma : I \to M$ on $M$ is said *calibrated* by $u$ if for all $[a,b] \subset I$, $\delta(u,\gamma_{|[a,b]}) = 0$.

**Remark 9.5.7.** We know from Proposition 9.5.4 that for all times $t < t'$ and points $q$ and $q'$ of $M$,

$$u(t',q') - u(t,q) \leq \inf \left\{ \int_t^{t'} L(\tau,\gamma(\tau),\dot\gamma(\tau))\,d\tau \;\middle|\; \begin{array}{rl} \gamma : & [t,t'] \to M \\ & s \mapsto q \\ & t \mapsto q' \end{array} \right\} \tag{9.5.7}$$

Hence, if $\gamma : [a,b] \to M$ is $u$-calibrated, then it is minimizing for the Lagrangian $L$.

**Proposition 9.5.8.** *1. If $\gamma : I \to M$ is a $u$-calibrated curve, then for all time $t$ in the interior of $I$, $u$ is differentiable at $(t,\gamma(t))$ and*

$$d_q u(t,\gamma(t)) = \partial_v L\big(t,\gamma(t),\dot\gamma(t)\big) \quad \text{and} \quad \mathscr{H}\big(t,\partial_t u(t,\gamma(t)),\gamma(t),d_q u(t,\gamma(t))\big) = 0 \tag{9.5.8}$$

*2. Let $A_{\varepsilon,u}$ be the set of points $(t,q) \in \mathbb{R}\times M$ such that there exists a curve $\gamma_{t,q}$ : $(t-\varepsilon,t+\varepsilon) \to M$ with $\gamma_{t,q}(t) = q$ which is $u$-calibrated. Then, the map*

$$\begin{aligned} A_{\varepsilon,u} &\longrightarrow T^*(\mathbb{R}\times M) \\ (t,q) &\longmapsto du(t,q) = \big(t,\partial_t u(t,q),q,d_q u(t,q)\big) \end{aligned}$$

*is locally Lipschitz.*

*Proof. 1. Differentiability.* Fix $(t,q) = (t,\gamma(t)) \in \mathbb{R}\times M$. We will bound $u$ in a neighbourhood of $(t,q)$ by two $C^1$ maps that coincide with it at this point. In order to do so, we use the domination inequality. Let $(s,y)$ be close enough to $(t,q)$ and fix two reference times $t^+ < t < t^-$ and $q^\pm = \gamma(t^\pm)$. From the calibration

$$u(t,q) = u(t^\pm,q^\pm) + \int_{t^\pm}^{t} L\big(\tau,\gamma(\tau),\dot\gamma(\tau)\big)\,d\tau =: \psi^\pm(t,q) \tag{9.5.9}$$

and from the domination

$$u(s,y) \le u(t^+,q^+) + \int_{t^+}^{t} L\big(\tau,\gamma^+_{(s,y)}(\tau),\dot\gamma^+_{(s,y)}(\tau)\big)\,d\tau =: \psi^+(s,y) \tag{9.5.10}$$

and

$$u(s,y) \ge u(t^-,q^-) - \int_{t}^{t^-} L\big(\tau,\gamma^-_{(s,y)}(\tau),\dot\gamma^-_{(s,y)}(\tau)\big)\,d\tau =: \psi^-(s,y) \tag{9.5.11}$$

where, in a chart around $(t,q)$

$$\gamma^{\pm}_{(s,y)}(\tau) = \gamma(\tau) + \frac{\tau - t^{\pm}}{s - t^{\pm}}(y - \gamma(s)) \tag{9.5.12}$$

are smooth families of curves linking $(t^{\pm},q^{\pm})$ to $(s,y)$ and such that $\gamma^{\pm}_{(t,q)} = \gamma$.
It is easy to see that $\psi^{\pm}$ are $C^1$. Moreover, $\psi^- \le u \le \psi^+$ with equalities at $(t,q)$. Then $u$ is $C^1$ at $(t,q)$.

*Evaluation of the Differential.* We differentiate (9.5.9) with respect to time $t$ without forgetting that $q = \gamma(t)$, to get

$$\partial_t u(t,\gamma(t)) + d_q u(t,\gamma(t)) = L\big(t,\gamma(t),\dot\gamma(t)\big) \tag{9.5.13}$$

And by Fenchel inequality (9.5.2) for $q = \gamma(t)$, $v = \dot\gamma(t)$ and $p = d_q u(t,\gamma(t))$, we have

$$\begin{aligned} 0 &= \partial_t u(t,\gamma(t)) + d_q u(t,\gamma(t)).\dot\gamma(t) - L\big(t,\gamma(t),\dot\gamma(t)\big) \\ &\le \partial_t u(t,q) + H(t,q,d_q u(t,q)) = \mathscr{H}\big(t,\partial_t u(t,\gamma(t)),\gamma(t),d_q u(t,\gamma(t))\big) \end{aligned} \tag{9.5.14}$$

In order to obtain the equality, we need to show that $\mathscr{H}\big(t,\partial_t u(t,\gamma(t)),\gamma(t),d_q u(t,\gamma(t))\big) \le 0$. This will follow from the convexity of the Hamiltonian $H$ on fibres and from a result due to Clarke (see [FM07] for a proof or [Cla83] for a more general result).

**Lemma 9.5.9.** *Let $f : U \to \mathbb{R}$ be a Lipschitz map defined on an open subset $U$ of $\mathbb{R}^d$ and let $U_0 \subset U$ be a subset of full Lebesgue measure. Let $q$ be a fixed element of $U$. We introduce the following sets*

$$\begin{aligned} K_f^{U_0}(q) &:= \left\{ \textit{limit points of } \big(df(q_n)\big)_{n\ge 0} \;\middle|\; q_n \in U_0,\ \lim_n q_n = q \right\} \subset T_q^*U \\ C_f^{U_0}(q) &:= \mathrm{Conv}(K_f^{U_0}(q)) \end{aligned} \tag{9.5.15}$$

*where* Conv *stands for the convex hull. Then, whenever $f$ is differentiable at a point $q \in U$, we have $df(q) \in C_f^{U_0}(q)$.*

We apply the lemma to the map $u : \mathbb{R} \times M \to \mathbb{R}$ and to the full measure subset $\mathcal{U}$ of

$\mathbb{R}\times M$ introduced in Proposition 9.4.20. We get that

$$du(t,\gamma(t)) = \big(\partial_t u(t,\gamma(t)), d_q u(t,\gamma(t))\big) \in C_u^{\mathcal{U}}(t,\gamma(t)) \tag{9.5.16}$$

Additionally, we know from the definition of the set $\mathcal{U}$ that for all $(s,q)\in\mathcal{U}$, $\big(s,\partial_t u(s,q),q,d_q u(s,q)\big)\subset \mathscr{L}\subset\{\mathscr{H}=0\}$. Hence, we deduce by continuity of $\mathscr{H}$ that

$$K_u^{\mathcal{U}}(t,\gamma(t)) \subset \{\mathscr{H}=0\}\cap T^*_{(t,\gamma(t))}(\mathbb{R}\times M) \quad\text{and}\quad C_u^{\mathcal{U}}(t,\gamma(t)) \subset \operatorname{Conv}\{\mathscr{H}=0\}\cap T^*_{(t,\gamma(t))}(\mathbb{R}\times M) \tag{9.5.17}$$

Moreover,

$$\{\mathscr{H}\le 0\}\cap T^*_{(t,\gamma(t))}(\mathbb{R}\times M) = \{(E,p)\in T^*_{(t,\gamma(t))}(\mathbb{R}\times M) \mid H(t,\gamma(t),p)\le -E\}$$

which corresponds, up to the symmetry $E\mapsto -E$, to the epigraph of the strictly convex Hamiltonian $H$ restricted to the fibre $T^*_{(t,\gamma(t))}(\mathbb{R}\times M)$. Thus, this set is strictly convex with boundary (or extremal points)

$$\{(E,p)\in T^*_{(t,\gamma(t))}(\mathbb{R}\times M) \mid H(t,\gamma(t),p) = -E\} = \{\mathscr{H}=0\}\cap T^*_{(t,\gamma(t))}(\mathbb{R}\times M)$$

Therefore, we deduce that $\operatorname{Conv}\{\mathscr{H}=0\}\cap T^*_{(t,\gamma(t))}(\mathbb{R}\times M) = \{\mathscr{H}\le 0\}\cap T^*_{(t,\gamma(t))}(\mathbb{R}\times M)$. Regrouping the inclusions (9.5.16) and (9.5.17), we obtain

$$du(t,\gamma(t)) \in \{\mathscr{H}\le 0\}\cap T^*_{(t,\gamma(t))}(\mathbb{R}\times M) \subset \{\mathscr{H}\le 0\}$$

and

$$\mathscr{H}\big(t,\partial_t u(t,\gamma(t)),\gamma(t),d_q u(t,\gamma(t))\big)\le 0 \tag{9.5.18}$$

This implies the equality all along the inequalities of (9.5.14) and (9.5.18) leading to the desired identities.

*2. $C^{1,1}$ Regularity.* We aim to use Fathi's criterion for a Lipschitz derivative, the proof of which can be found in proposition 4.11.3 of [Fat08].

**Lemma 9.5.10.** *Fix a point $q_0$ of $\mathbb{R}^d$ and a radius $r>0$. Let $u: B(q_0,r)\to\mathbb{R}$ be a function and let $C>0$ be a positive constant. We introduce the set $A^{C,u}$ of $B(q_0,r)$ as*

$$A^{C,u} = \left\{q\in B(q_0,r) \mid \exists\varphi_q:\mathbb{R}^d\to\mathbb{R} \text{ linear, } \forall y\in B(q_0,r),\ \|u(y)-u(q)-\varphi_q(y-q)\|\le C\|y-q\|^2\right\}$$

*Then $u$ has for all $q\in A^{C,u}$, $d_q u=\varphi_q$ and the restriction of $q\mapsto d_q u$ to $\{q\in A^{C,u} \mid \|q-q_0\|\le r/3\}$ is Lipschitz with Lipschitz constant $6C$.*

We will search for a constant $C$ such that the elements $A_{\varepsilon,u}\subset A^{C,u}$. Keeping the notation of the previous part of the proof, we set $t^\pm = t\pm\varepsilon$ and we consider a neighbourhood

$[s^+, s^-] \times V$ of $(t, q)$ such that the curves $\gamma^{\pm}_{(s,y)}(\tau)$ remains in the chart we are working in for all $\tau \in [t^+, t^-]$. We take $t^+ + \varepsilon/2 < s^+ < t < s^- < t^- - \varepsilon/2$ and $V \subset B(q, \varepsilon)$.

We have $\psi^- \leq u \leq \psi^+$ and $\psi^-(t, q) = u(t, q) = \psi^+(t, q)$. Thus, for $(s, y) \in (t^-, t^+) \times M$,

$$\psi^-(s, y) - \psi^-(t, q) \leq u(s, y) - u(t, q) \leq \psi^+(s, y) - \psi^+(t, q) \tag{9.5.19}$$

We estimate the right-hand side by applying a 2 Taylor expansion on the map $\psi^+$ at $(t, q)$.

$$\left|\psi^+(s, y) - \psi^+(t, q) - d\psi^+(t, q).(s - t, y - q)\right| \leq \left\|d^2\psi^+\right\|_\infty^{[s^+,s^-]\times V} .\left(|s - t|^2 + \|y - q\|^2\right) \tag{9.5.20}$$

where $\|\cdot\|_\infty^{[s^+,s^-]\times V}$ stands for the $\|\cdot\|_\infty$-norm of the restriction to the set $[s^+, s^-] \times V$. We set

$$C' = \left\|d^2\psi^+\right\|_\infty^{[s^+,s^-]\times V} \tag{9.5.21}$$

We would like to see that it is independent of $(t, q)$. This is given by the following classical lemma first proved by John N. Mather in [Mat91].

**Lemma 9.5.11.** *(A Priori Compactness) Let $L : TM \to \mathbb{R}$ be a Tonelli Lagrangian and fix a small positive $\varepsilon > 0$. Then, there exists a compact subset $K_\varepsilon$ of $TM$ such that every minimizing curve $\gamma : [s, t] \to M$ with $t - s \geq \varepsilon$ verifies $(\gamma(\tau), \dot\gamma(\tau)) \in K_\varepsilon$.*

Since the curve $\gamma$ is minimizing and of time length $2\varepsilon$, we infer from the à priori compactness that $(\gamma, \dot\gamma)$ is contained in the compact set $K_{2\varepsilon}$. Moreover, since the minimizing curves follow the Lagrangian flow $\phi_L$ of $L$, we have $\ddot\gamma(\tau) = dv \circ X_L(\tau, \gamma(\tau), \dot\gamma(\tau))$ so that $\{\ddot\gamma(\tau) \mid \tau \in [t^+, t^-]\}$ is contained in a compact set that only depends on $\varepsilon$. Consequently, the set $\{(\gamma^{\pm}_{(s,y)}(\tau), \dot\gamma^{\pm}_{(s,y)}(\tau), \ddot\gamma^{\pm}_{(s,y)}(\tau)) \mid (\tau, s, y) \in [t^+, t^-] \times [s^+, s^-] \times V\}$ is contained in a compact set $K$ of $T^*M$ independent of $(t, q)$. Moreover, we took $[t^+, t^-] \times [s^+, s^-] \times V \subset [t - \varepsilon, t + \varepsilon] \times [t - \varepsilon/2, t + \varepsilon/2] \times B(q, \varepsilon)$. This shows that the constant $C'$ only depends on $\varepsilon$.

Let us now evaluate $d\psi^+(s, y)$ where we recall the definition (9.5.10) of $\psi^+$. We have

$$\begin{aligned}
\partial_s\psi^+(t, q) &= \int_{t^+}^{t} \partial_q L(\tau, \gamma(\tau), \dot\gamma(\tau)).\partial_s\gamma^+_{(t,q)}(\tau)\, d\tau \\
&\quad + \int_{t^+}^{t} \partial_v L(\tau, \gamma(\tau), \dot\gamma(\tau)).\partial_s\dot\gamma^+_{(t,q)}(\tau)\, d\tau + L(t, \gamma(t), \dot\gamma(t)) \\
d_y\psi^+(t, q) &= \int_{t^+}^{t} \partial_q L(\tau, \gamma(\tau), \dot\gamma(\tau)).d_y\gamma^+_{(t,q)}(\tau)\, d\tau \\
&\quad + \int_{t^+}^{t} \partial_v L(\tau, \gamma(\tau), \dot\gamma(\tau)).d_y\dot\gamma^+_{(t,q)}(\tau)\, d\tau
\end{aligned} \tag{9.5.22}$$

so that

$$\begin{aligned}
d\psi^+(t,q).(s-t,y-q) &= \partial_s\psi^+(t,q).(s-t) + d_y\psi^+(t,q).(y-q) \\
&= \int_{t^+}^{t} \partial_q L(\tau,\gamma(\tau),\dot\gamma(\tau)).\big[\partial_s\gamma^+_{(t,q)}(\tau).(s-t) + d_y\gamma^+_{(t,q)}(\tau).(y-q)\big]\,d\tau \\
&+ \int_{t^+}^{t} \partial_v L(\tau,\gamma(\tau),\dot\gamma(\tau)).\big[\partial_s\dot\gamma^+_{(t,q)}(\tau).(s-t) + d_y\dot\gamma^+_{(t,q)}(\tau).(y-q)\big]\,d\tau \\
&+ L(t,\gamma(t),\dot\gamma(t)).(s-t)
\end{aligned} \tag{9.5.23}$$

We evaluate the differential of $\gamma^+_{(s,y)}$ and $\dot\gamma^+_{(s,y)}$ with respect to $s$ and $y$. Recall that

$$\gamma^+_{(s,y)}(\tau) = \gamma(\tau) + \frac{\tau - t^+}{s-t^+}(y-\gamma(s)) \quad \text{and} \quad \dot\gamma^+_{(s,y)}(\tau) = \dot\gamma(\tau) + \frac{1}{s-t^+}(y-\gamma(s))$$

so that

$$\begin{aligned}
\partial_s\gamma^+_{(s,y)}(\tau) &= -\frac{\tau-t^+}{(s-t^+)^2}(y-\gamma(s)) - \frac{\tau-t^+}{s-t^+}\dot\gamma(s), & \partial_s\gamma^+_{(t,q)}(\tau) &= -\frac{\tau-t^+}{t-t^+}\dot\gamma(t) \\
d_y\gamma^+_{(s,y)}(\tau) &= \frac{\tau-t^+}{s-t^+}dy, & d_y\gamma^+_{(t,q)}(\tau).(y-q) &= \frac{\tau-t^+}{t-t^+}(y-q)
\end{aligned}$$

and

$$\begin{aligned}
\partial_s\dot\gamma^+_{(s,y)}(\tau) &= -\frac{1}{(s-t^+)^2}(y-\gamma(s)) - \frac{1}{s-t^+}\dot\gamma(s), & \partial_s\dot\gamma^+_{(t,q)}(\tau) &= -\frac{1}{t-t^+}\dot\gamma(t) \\
d_y\dot\gamma^+_{(s,y)}(\tau) &= \frac{1}{s-t^+}dy, & d_y\dot\gamma^+_{(t,q)}(\tau).(y-q) &= \frac{1}{t-t^+}(y-q)
\end{aligned}$$

Hence, we get

$$\begin{aligned}
&\partial_q L(\tau,\gamma(\tau),\dot\gamma(\tau)).\big[\partial_s\gamma^+_{(t,q)}(\tau).(s-t) + \partial_y\gamma^+_{(t,q)}(\tau).(y-q)\big] \\
&\qquad = \partial_q L(\tau,\gamma(\tau),\dot\gamma(\tau)).\big[-\frac{\tau-t^+}{t-t^+}(s-t).\dot\gamma(t) + \frac{\tau-t^+}{t-t^+}(y-q)\big] \\
&\qquad\quad = (\tau-t^+).\partial_q L(\tau,\gamma(\tau),\dot\gamma(\tau)).\big[-\frac{1}{t-t^+}(s-t).\dot\gamma(t) + \frac{1}{t-t^+}(y-q)\big]
\end{aligned}$$

and

$$\begin{aligned}
&\partial_v L(\tau,\gamma(\tau),\dot\gamma(\tau)).\big[\partial_s\dot\gamma^+_{(t,q)}(\tau).(s-t) + \partial_y\dot\gamma^+_{(t,q)}(\tau).(y-q)\big] \\
&\qquad\qquad = \partial_v L(\tau,\gamma(\tau),\dot\gamma(\tau)).\big[-\frac{1}{t-t^+}(s-t).\dot\gamma(t) + \frac{1}{t-t^+}(y-q)\big]
\end{aligned}$$

And since the curve $\gamma$ is calibrated, it is minimizing and it verifies the Euler-Lagrange equation (see proposition 2.2.6 of [Fat08])

$$\begin{aligned}(\tau-t^+).\partial_q L(\tau,\gamma(\tau),\dot\gamma(\tau))+\partial_v L(\tau,\gamma(\tau),\dot\gamma(\tau)) &= (\tau-t^+).\frac{d}{d\tau}\big(\partial_v L(\tau,\gamma(\tau),\dot\gamma(\tau))\big)+\partial_v L(\tau,\gamma(\tau),\dot\gamma(\tau))\\ &= \frac{d}{d\tau}\Big((\tau-t^+).\partial_v L(\tau,\gamma(\tau),\dot\gamma(\tau))\Big)\end{aligned}$$

Going back to identity (9.5.23), we have shown that

$$\begin{aligned}d\psi^+(t,q).(s-t,y-q) &= (t-t^+).\partial_v L(t,\gamma(t),\dot\gamma(t)).\big[-\frac{1}{t-t^+}(s-t).\dot\gamma(t)+\frac{1}{t-t^+}(y-q)\big]\\ &\quad + L(t,\gamma(t),\dot\gamma(t)).(s-t)\\ &= \big[-\partial_v L(t,\gamma(t),\dot\gamma(t)).\dot\gamma(t)+L(t,\gamma(t),\dot\gamma(t))\big].(s-t)+\partial_v L(t,\gamma(t),\dot\gamma(t)).(y-q)\\ &= -H\big(t,q,\partial_v L(t,q,\dot\gamma(t))\big).(s-t)+\partial_v L(\tau,q,\dot\gamma(t)).(y-q)\end{aligned} \tag{9.5.24}$$

where we last used the equality case of the Fenchel inequality (9.5.2).

Gathering (9.5.20), (9.5.21) and (9.5.24), we obtain

$$\psi^+(s,y)-\psi^+(t,q)\le -H\big(t,q,\partial_v L(t,q,\dot\gamma(t))\big).(s-t)+\partial_v L(\tau,q,\dot\gamma(t)).(y-q)+C'\big(|s-t|^2+\|y-q\|^2\big)$$

Analogously for $\psi^-$, we find a constant $C''>0$ depending only on $\varepsilon$ such that

$$\psi^-(s,y)-\psi^-(t,q)\ge -H\big(t,q,\partial_v L(t,q,\dot\gamma(t))\big).(s-t)+\partial_v L(\tau,q,\dot\gamma(t)).(y-q)-C''\big(|s-t|^2+\|y-q\|^2\big)$$

And we finally get from (9.5.19) that

$$\left|u(s,y)-u(t,q)+H\big(t,q,\partial_v L(t,q,\dot\gamma(t))\big).(s-t)-\partial_v L(\tau,q,\dot\gamma(t)).(y-q)\right|\le C\big(|s-t|^2+\|y-q\|^2\big)$$

This allows to apply Lemma 9.5.10 and to conclude that

$$d_q u(t,\gamma(t))=\partial_v L\big(t,\gamma(t),\dot\gamma(t)\big),\quad \partial_t u(t,q)=-H(t,q,d_q u(t,q))$$

and that the map $(t,q)\longmapsto du(t,q)=\big(t,\partial_t u(t,q),q,d_q u(t,q)\big)$ restricted to $A_{\varepsilon,u}$ is locally Lipschitz. □

### 9.5.2 Setting and Notations

We keep the same notations as in the statement of Theorem 9.1.5.

Let $\mathcal{L}$ be a Lagrangian submanifold of $T^*M$, $H$-isotopic to the zero section $0_{T^*M}$, such that there exist two increasing sequences of integers $n_k$ and $m_k$ such that $(\mathcal{L}_{n_k})_{k\ge0}=(\phi_H^{n_k}(\mathcal{L}))_{k\ge0}$ and $(\mathcal{L}_{-m_k})_{k\ge0}=(\phi_H^{-m_k}(\mathcal{L}))_{k\ge0}$ respectively converge with reduced complexities to Lagrangian submanifolds $\mathcal{L}_\omega$ and $\mathcal{L}_\alpha$ which are $H$-isotopic to the zero section $0_{T^*M}$.

According to the Definition 9.1.2 of reduced complexity convergence, if we consider two Hamiltonian maps $\varphi_\alpha$ and $\varphi_\omega \in \mathrm{Ham}(T^*M, \omega)$ such that $\mathcal{L}_\alpha = \varphi_\alpha(0_{T^*M})$ and $\mathcal{L}_\omega = \varphi_\omega(0_{T^*M})$, then we have

i. $(\mathcal{L}_{-m_k})_{k\geq 0}$ and $(\mathcal{L}_{n_k})_{k\geq 0}$ converge respectively to $\mathcal{L}_\alpha$ and $\mathcal{L}_\omega$ in the Haussdorff topology.

ii. if we denote by $l_k^\alpha$ and $l_k^\omega$ respective Liouville primitives on $\varphi_\alpha^{-1}(\mathcal{L}_{-m_k})$ and $\varphi_\omega^{-1}(\mathcal{L}_{n_k})$, then we have $\lim_k \mathrm{Osc}(l_k^\alpha) = \lim_k \mathrm{Osc}(l_k^\omega) = 0$.

We choose the Liouville primitives $l_k^\alpha$ and $l_k^\omega$ on $\varphi_\alpha^{-1}(\mathcal{L}_{-m_k})$ and $\varphi_\omega^{-1}(\mathcal{L}_{n_k})$ as follows. The Hamiltonian maps $\varphi_\alpha$ and $\varphi_\omega$ are by definition exact symplectomorphisms of $(T^*M, \lambda)$ and there exist regular maps $f_\alpha$ and $f_\omega : T^*M \to \mathbb{R}$ such that

$$\varphi_\alpha^*\lambda - \lambda = df_\alpha \quad \text{and} \quad \varphi_\omega^*\lambda - \lambda = df_\omega \tag{9.5.25}$$

Then, for all integer $k \in \mathbb{N}$, we choose the Liouville primitives following Lemma 9.1.14.

$$l_k^\alpha = h_{-m_k} \circ \varphi_\alpha - f_\alpha \quad \text{and} \quad l_k^\omega = h_{n_k} \circ \varphi_\omega - f_\omega \tag{9.5.26}$$

As in Section 9.2, we extend $\mathcal{L}_\alpha$ and $\mathcal{L}_\omega$ respectively to exact Lagrangian submanifolds $\mathscr{L}_\alpha$ and $\mathscr{L}_\omega$ of $T^*\mathcal{M} = T^*(\mathbb{R}\times M)$. And we can successively construct :

1. Liouville primitives $h_\alpha : \mathcal{L}_\alpha \to \mathbb{R}$ for $\mathcal{L}_\alpha$ and $h_\omega : \mathcal{L}_\omega \to \mathbb{R}$ for $\mathcal{L}_\omega$, with time evolution $h_t^\alpha$ and $h_t^\omega$ following (9.3.3).

2. Liouville primitives $\hbar_\alpha : \mathscr{L}_\alpha \to \mathbb{R}$ for $\mathscr{L}_\alpha$ and $\hbar_\omega : \mathscr{L}_\omega \to \mathbb{R}$ for $\mathscr{L}_\omega$ with

$$\hbar_\alpha(t, E_\alpha, q, p_\alpha) = h_{\alpha,t}(q, p_\alpha) \quad \text{and} \quad \hbar_\omega(t, E_\omega, q, p_\omega) = h_{\omega,t}(q, p_\omega) \tag{9.5.27}$$

   whenever $(t, E_\alpha, q, p_\alpha) \in \mathscr{L}_\alpha$ and $(t, E_\omega, q, p_\omega) \in \mathscr{L}_\omega$.

3. Graph selectors $u_\alpha : \mathbb{R}\times M \to \mathbb{R}$ for $\mathscr{L}_\alpha$ and $u_\omega : \mathbb{R}\times M \to \mathbb{R}$ for $\mathscr{L}_\omega$ respectively associated with $\hbar_\alpha$ and $\hbar_\omega$, following Subsection 9.4.3. Proposition 9.4.20 provides two open full-measure subsets $\mathcal{U}_\alpha$ and $\mathcal{U}_\omega$ of $\mathbb{R}\times M$ where $u_\alpha$ and $u_\omega$ are respectively regular.

Note that the Liouville primitives $l^\alpha = h_\alpha \circ \varphi_\alpha - f_\alpha$ and $l^\omega = h_\omega \circ \varphi_\omega - f_\omega$ of the zero section $\varphi_\alpha^{-1}(\mathcal{L}_\alpha) = \varphi_\omega^{-1}(\mathcal{L}_\omega) = 0_{T^*M}$ are constant. In this case, we have

$$\varphi_\alpha^{-1}(\mathcal{L}_{n_k})\#\overline{\varphi_\alpha^{-1}(\mathcal{L}_\alpha)} = \varphi_\alpha^{-1}(\mathcal{L}_{n_k}) \quad \text{and} \quad \varphi_\omega^{-1}(\mathcal{L}_{n_k})\#\overline{\varphi_\omega^{-1}(\mathcal{L}_\omega)} = \varphi_\omega^{-1}(\mathcal{L}_{n_k})$$

with associated Liouville primitive $l_k^\alpha$ and $l_k^\omega$. This gives a meaning to $\mathrm{Osc}(l_k^\alpha - l^\alpha) = \mathrm{Osc}(l_k^\alpha)$ and $\mathrm{Osc}(l_k^\omega - l^\omega) = \mathrm{Osc}(l_k^\omega)$ where $l^\alpha$ and $l^\omega$ are considered to be elements of $\mathbb{R}$.

### 9.5.3 Study of Limit Points

Let us begin the proof of the main Theorem 9.1.5. The most crucial step is dealt with in this subsection. The idea is to initially identify curves for which calibration can be easily established.

Let $x = (q,p)$ be a point of $\mathcal{L}$ and $x_\omega = (q_\omega, p_\omega)$ be a limit point of the sequence $\left(\phi_H^{n_k}(x)\right)_k$. We know from the Hausdorff convergence of $\mathcal{L}_{n_k}$ that the point $x_\omega$ belongs to $\mathcal{L}_\omega$. And set $x_\omega(t) = (q_\omega(t), p_\omega(t)) = \phi_H^t(x_\omega)$. We can assume, up to extraction, that

$$\phi_H^{n_k}(x) \longrightarrow x_\omega \quad \text{and} \quad n_{k+1} - n_k \longrightarrow +\infty \quad \text{as } k \to \infty \tag{9.5.28}$$

Similarly, let $x_\alpha = (q_\omega, p_\omega) \in \mathcal{L}_\alpha$ be a limit point of the sequence $\left(\phi_H^{-m_k}(x)\right)_k$. Set $x_\alpha(t) = (q_\alpha(t), p_\alpha(t)) = \phi_H^t(x_\alpha)$. And assume, up to extraction, that

$$\phi_H^{-m_k}(x) \longrightarrow x_\alpha \quad \text{and} \quad m_{k+1} - m_k \longrightarrow +\infty \quad \text{as } k \to \infty \tag{9.5.29}$$

**Proposition 9.5.12.** *The curves $q_\alpha(t)$ and $q_\omega(t)$ are respectively calibrated by $u_\alpha$ and $u_\omega$.*

*Proof.* We only prove the calibration for $q_\omega(t)$. The case of $q_\alpha(t)$ is done analogously. Let $a < b$ be two times and $b_k = a + n_{k+1} - n_k$. Set $u_k(t,q) = u(t+n_k, q)$, $x_k = (q_k, p_k) = \phi_H^{n_k}(x)$ and $x_k(t) = (q_k(t), p_k(t)) = \phi_H^t(x_k)$. In order to compute the calibration defect $\delta(u_\omega, q_{\omega|[a,b]})$, we need to link it to

$$\delta(u_k, q_{k|[a,b]}) = \int_a^b L(s, q_k(s), \dot{q}_k(s))\, ds - [u_k(b, q_k(b)) - u_k(a, q_k(a))]$$

This requires to know more on the asymptotic $C^1$ convergence of the curve $q_k$ and on the $C^0$ convergence of maps $u_k$ restricted to $[a,b] \times M$.

**Lemma 9.5.13.** *We have*

$$\delta(u_\omega, q_{\omega|[a,b]}) = \lim_k \delta(u_k, q_{k|[a,b]})$$

*Proof.* Since the points $x_k = (q_k, p_k)$ converge to $x_\omega = (q_\omega, p_\omega)$, the curves $x_{k|[a,b]}$ converge uniformly to $x_{\omega|[a,b]}$. We also know from (9.5.2) that $p_k(t) = \partial_v L(t, q_k(t), \dot{q}_k(t))$ and $p_\omega(t) = \partial_v L(t, q_\omega(t), \dot{q}_\omega(t))$. Thus the curves $q_{k|[a,b]}$ converge to $q_{\omega|[a,b]}$ in the $C^1$ topology so that

$$\lim_k \int_a^b L(s, q_k(s), \dot{q}_k(s))\, ds = \int_a^b L(s, q_\omega(s), \dot{q}_\omega(s))\, ds \tag{9.5.30}$$

For the comparison between the $u$ maps, we have for all time $t$, $u_k(t)$ and $u_\omega(t)$ are respective graph selectors of the Lagrangian submanifolds $\mathcal{L}_{t+n_k}$ and $\mathcal{L}_{\omega+t} := \phi_H^t(\mathcal{L}_\omega)$. Then, applying Proposition 9.4.18, we have

$$c(1, \mathcal{L}_{t+n_k} \# \overline{\mathcal{L}_{\omega+t}}) \leq u_k(t) - u_\omega(t) \leq c(\mu, \mathcal{L}_{t+n_k} \# \overline{\mathcal{L}_{\omega+t}}) \tag{9.5.31}$$

and an application of Propositions 9.4.11 and 9.4.4 gives

$$c(1, \mathcal{L}_{t+n_k} \# \overline{\mathcal{L}_{\omega+t}}) = c(1, \mathcal{L}_{n_k} \# \overline{\mathcal{L}_\omega}) = c(1, \varphi_\omega^{-1}(\mathcal{L}_{n_k}) \# \overline{\varphi_\omega^{-1}(\mathcal{L}_\omega)}) \geq \min(l_k^\omega - l^\omega) \tag{9.5.32}$$

and

$$c(\mu, \mathcal{L}_{t+n_k} \# \overline{\mathcal{L}_{\omega+t}}) = c(\mu, \varphi_\omega^{-1}(\mathcal{L}_{n_k}) \# \overline{\varphi_\omega^{-1}(\mathcal{L}_\omega)}) \leq \max(l_k^\omega - l^\omega) \tag{9.5.33}$$

where $l^\omega$ is the constant Liouville primitive on $\varphi_\omega^{-1}(\mathcal{L}_\omega) = 0_{T^*M}$. Gathering these three inequalities (9.5.31), (9.5.32) and (9.5.33), we obtain for all $(t, q) \in \mathbb{R} \times M$

$$\min(l_k^\omega - l^\omega) \leq u_k(t, q) - u_\omega(t, q) \leq \max(l_k^\omega - l^\omega) \tag{9.5.34}$$

Applying these inequalities at $t = a$ and $t = b$ yields

$$\left|[u_k(b, q_k(b)) - u_\omega(b, q_k(b))] - [u_k(a, q_k(a)) - u_\omega(a, q_k(a))]\right| \leq \operatorname{Osc}(l_k^\omega) \longrightarrow 0 \quad \text{as } k \to \infty \tag{9.5.35}$$

Hence, we get

$$\begin{aligned}
&\left|[u_k(b, q_k(b)) - u_k(a, q_k(a))] - [u_\omega(b, q_\omega(b)) - u_\omega(a, q_\omega(a))]\right| \\
&\quad \leq \left|[u_k(b, q_k(b)) - u_k(a, q_k(a))] - [u_\omega(b, q_k(b)) - u_\omega(a, q_k(a))]\right| \\
&\quad + \left|u_\omega(b, q_\omega(b)) - u_\omega(b, q_k(b))\right| + \left|u_\omega(a, q_\omega(a)) - u_\omega(a, q_k(a))\right| \\
&\quad \leq \operatorname{Osc}(l_k^\omega) + \left|u_\omega(b, q_\omega(b)) - u_\omega(b, q_k(b))\right| + \left|u_\omega(a, q_\omega(a)) - u_\omega(a, q_k(a))\right|
\end{aligned} \tag{9.5.36}$$

Moreover, we know from the convergence of the curves $q_{k|[a,b]}$ to $q_{\omega|[a,b]}$ and the continuity of $u_\omega$ that

$$\lim_k u_\omega(b, q_k(b)) = u_\omega(b, q_\omega(b)) \quad \text{and} \quad \lim_k u_\omega(a, q_k(a)) = u_\omega(a, q_\omega(a)) \tag{9.5.37}$$

Hence, we deduce from (9.5.36) and (9.5.37) that

$$\lim_k u_k(b, q_k(b)) - u_k(a, q_k(a)) = u_\omega(b, q_\omega(b)) - u_\omega(a, q_\omega(a)) \tag{9.5.38}$$

Gathering (9.5.30) and (9.5.38), we conclude that

$$\lim_k \delta(u_k, q_{k|[a,b]}) = \lim_k \int_a^b L(s, q_k(s), \dot{q}_k(s))\, ds - \lim_k [u_k(b, q_k(b)) - u_k(a, q_k(a))]$$

$$= \int_a^b L(s, q_\omega(s), \dot{q}_\omega(s))\, ds + [u_\omega(b, q_\omega(b)) - u_\omega(a, q_\omega(a))] = \delta(u_\omega, q_{\omega|[a,b]})$$

□

From assumption (9.5.28), we have inclusion $[a,b] \subset [a, b_k]$ for large $k$. Thus, the lemma and the positivity of the defect of calibration (Proposition 9.5.4) lead to

$$0 \leq \delta(u_\omega, q_{\omega|[a,b]}) = \lim_k \delta(u_k, q_{k|[a,b]}) \leq \liminf_k \delta(u_k, q_{k|[a,b_k]}) \tag{9.5.39}$$

Let us evaluate

$$\delta(u_k, q_{k|[a,b_k]}) = \int_a^{b_k} L(s, q_k(s), \dot{q}_k(s))\, ds - [u_k(b_k, q_k(b_k)) - u_k(a, q_k(a))]$$

From the Fenchel's equality case, we know that for all time $s \in \mathbb{R}$,

$$L(s + n_k, q_k(s), \dot{q}_k(s)) = p_k(s).\dot{q}_k(s) - H(s + n_k, q_k(s), p_k(s)) \tag{9.5.40}$$

Set $\zeta_k(t) = (t + n_k, E_k(t), q_k(t), p_k(t))$ to be a curve in $\mathscr{L}$. Since $\mathscr{L} \subset \{\mathscr{H} = 0\}$ ,We have

$$E_k(t) = -H\big(t + n_k, q_k(t), p_k(t)\big) \tag{9.5.41}$$

Therefore, using the time one periodicity of $L$, we get

$$\begin{aligned}
\int_a^{b_k} L(s, q_k(s), \dot{q}_k(s))\, ds &= \int_a^{b_k} L(s + n_k, q_k(s), \dot{q}_k(s))\, ds \\
&= \int_a^{b_k} p_k(s).\dot{q}_k(s) - H(s + n_k, q_k(s), p_k(s))\, ds \\
&= \int_a^{b_k} p_k(s).\dot{q}_k(s) + E_k(s)\, ds = \int_\zeta \Lambda_{|\mathscr{L}} = \int_\zeta d\hbar \\
&= \hbar(\zeta(b_k)) - \hbar(\zeta(a)) = h_{a+n_{k+1}}(x_k(b_k)) - h_{a+n_k}(x_k(a))
\end{aligned}$$

To simplify the notation, set $x_k^a = (q_k^a, p_k^a) = x_k(a)$ and note that

$$x_k(b_k) = x_k(a + n_{k+1} - n_k) = \phi_H^{a+n_{k+1}-n_k}(x_k) = \phi_H^a(x_{k+1}) = x_{k+1}^a$$

and

$$u_k(b_k) = u(b_k + n_k) = u(a + n_{k+1}) = u_{k+1}(a)$$

Hence

$$\delta(u_k, q_{k|[a,b_k]}) = [h_{a+n_{k+1}}(x_{k+1}^a) - h_{a+n_k}(x_k^a)] - [u_{k+1}(a, q_{k+1}^a) - u_k(a, q_k^a)] \tag{9.5.42}$$

We insert the terms $h^\omega_a(x^a_\omega)$ and $u_\omega(a, q^a_\omega)$ as follows

$$\begin{aligned}\delta(u_k, q_{k|[a,b_k]}) = [h_{a+n_{k+1}}(x^a_{k+1}) - h^\omega_a(x^a_\omega)] - [u_{k+1}(a, q^a_{k+1}) - u_\omega(a, q^a_\omega)] \\ + [h^\omega_a(x^a_\omega) - h_{a+n_k}(x^a_k)] - [u_\omega(a, q^a_\omega) - u_k(a, q^a_k)]\end{aligned} \tag{9.5.43}$$

We need to prove that the sequence

$$\delta_k = [h_{a+n_k}(x^a_k) - h^\omega_a(x^a_\omega)] - [u_k(a, q^a_k) - u_\omega(a, q^a_\omega)] \tag{9.5.44}$$

converges to 0. This involves comparing $u_{k+1} - u_k$ and "$h_{a+n_{k+1}} - h_{a+n_k}$" where the second term does not make sense, justifying the need to use the primitives $l_t$ introduced in (9.5.26).

Let $c_k$ be any value in the image of $u_k(a) - u_\omega(a)$. We infer from the inequalities (9.5.34) that

$$\|l^\omega_k - l^\omega - c_k\|_\infty \leq Osc(l^\omega_k) \quad \text{and} \quad \|u_k(a) - u_\omega(a) - c_k\|_\infty \leq Osc(u_k(a) - u_\omega(a)) \leq Osc(l^\omega_k) \tag{9.5.45}$$

We get

$$|\delta_k| \leq |h_{a+n_k}(x^a_k) - h^\omega_a(x^a_\omega) - c_k| + |u_k(a, q^a_k) - u_\omega(a, q^a_\omega) - c_k| \tag{9.5.46}$$

The choice of the constant $c_k$ leads to

$$\begin{aligned}|u_k(a, q^a_k) - u_\omega(a, q^a_\omega) - c_k| &\leq |u_k(a, q^a_\omega) - u_\omega(a, q^a_\omega) - c_k| + |u_\omega(a, q^a_\omega) - u_\omega(a, q^a_k)| \\ &\leq Osc(u_k(a) - u_\omega(a)) + |u_\omega(a, q^a_\omega) - u_\omega(a, q^a_k)| \\ &\leq Osc(l_k) + |u_\omega(a, q^a_\omega) - u_\omega(a, q^a_k)| \longrightarrow 0 \quad \text{ as } k \to \infty\end{aligned}$$

where we used the continuity of $u_\omega$. For the remaining term of (9.5.46), we first use the identity (9.3.6) on $h$ and $h^\omega$ to get

$$\begin{aligned}|h_{a+n_k}(x^a_k) - h^\omega_a(x^a_\omega) - c_k| \leq \left| \int_0^a (x^*_k\lambda - x^*_\omega\lambda) - \int_0^a H(\tau, x_k(\tau)) - H(\tau, x_\omega(\tau))\, d\tau \right| \\ + |h_{n_k}(x_k) - h^\omega_0(x_\omega) - c_k|\end{aligned}$$

Since the curve $x_k$ converges to $x_\omega$ uniformly on $[0, a]$, and since by the identity (9.5.3) the curve $\dot{q}_k$ converges uniformly to $q_\omega$ on the same interval $[0, a]$, we infer that

$$\left| \int_0^a (x^*_k\lambda - x^*_\omega\lambda) - \int_0^a H(\tau, x_k(\tau)) - H(\tau, x_\omega(\tau))\, d\tau \right| \longrightarrow 0 \quad \text{ as } k \to \infty$$

Set $\tilde{x}_k = \varphi_\omega^{-1}(x_k)$ and $\tilde{x}_\omega = \varphi_\omega^{-1}(x_\omega)$. Then, applying (9.5.26) yields

$$\begin{aligned}|h_{n_k}(x_k) - h_0^\omega(x_\omega) - c_k| &\leq |l_k^\omega(\tilde{x}_k) - l^\omega - c_k| + |f_\omega(\tilde{x}_k) - f_\omega(\tilde{x}_\omega)| \\ &\leq Osc(l^\omega) + |f_\omega(\tilde{x}_k) - f_\omega(\tilde{x}_\omega)| \longrightarrow 0 \quad \text{as } k \to \infty\end{aligned}$$

where we used the continuity of the map $f_\omega$. This establishes the limit $\lim_k \delta_k = 0$ and hence

$$0 \leq \delta(u_\omega, q_{|[a,b]}) \leq \liminf_k \delta(u_k, q_{k|[a,b_k]}) = \liminf_k (\delta_{k+1} - \delta_k) = 0$$

□

### 9.5.4 Study of all Points

We now extend the previous $u_\omega$-calibration result to the $u$-calibration of every Hamiltonian curve included in $\mathscr{L}$.

**Proposition 9.5.14.** *Let $x = (q,p)$ be a point of $\mathcal{L}$ and $x(t) = (q(t), p(t)) = \phi_H^t(x)$. Then the curve $q(t)$ is calibrated by $u$.*

In the proof of the Main Theorem 9.1.5, we will see that this calibration, by Fathi's result 9.5.8 on calibrated curves, ensures that the submanifolds $\mathcal{L}_t$ are graphs over the zero section of $T^*M$ for all times $t \in \mathbb{R}$.

*Proof.* We use the notation $x_t = (q_t, p_t)$ instead of $x(t) = (q(t), p(t))$. Let $a < b$ be two real times. We will estimate the defect of calibration $\delta(u, q_{|[a,b]})$. But first, let $x_\alpha = (q_\alpha, p_\alpha) \in \mathcal{L}_\alpha$ and $x_\omega = (q_\omega, p_\omega) \in \mathcal{L}_\omega$ be respective limit points of the sequences $\left(\phi_H^{-m_k}(x)\right)_{k \geq 0}$ and $\left(\phi_H^{n_k}(x)\right)_{k \geq 0}$. For $k$ large enough, $[a,b] \subset [-m_k, n_k]$, hence

$$0 \leq \delta(u, q_{|[a,b]}) \leq \liminf_k \delta(u, q_{|[-m_k,n_k]}) \tag{9.5.47}$$

and from the same computation that led to (9.5.42), we get

$$\begin{aligned}\delta(u, q_{|[-m_k,n_k]}) &= [h_{n_k}(x_{n_k}) - h_{-m_k}(x_{-m_k})] - [u(n_k, q_{n_k}) - u(-m_k, q_{-m_k})] \\ &= [h_{n_k}(x_{n_k}) - u(n_k, q_{n_k})] - [h_{-m_k}(x_{-m_k}) - u(-m_k, q_{-m_k})]\end{aligned} \tag{9.5.48}$$

Let $\delta_\omega^k$ and $\delta_\alpha^k$ be defined as

$$\begin{aligned}\delta_\alpha^k &= [h_{-m_k}(x_{-m_k}) - u(-m_k, q_{-m_k})] - [h_0^\alpha(x_\alpha) - u_\alpha(0, q_\alpha)] \\ &= [h_{-m_k}(x_{-m_k}) - h_0^\alpha(x_\alpha)] - [u(-m_k, q_{-m_k}) - u_\alpha(0, q_\alpha)]\end{aligned}$$

and

$$\begin{aligned}\delta_\omega^k &= [h_{n_k}(x_{n_k}) - u(n_k, q_{n_k})] - [h_0^\omega(x_\omega) - u_\omega(0, q_\omega)] \\ &= [h_{n_k}(x_{n_k}) - h_0^\omega(x_\omega)] - [u(n_k, q_{n_k}) - u_\omega(0, q_\omega)]\end{aligned}$$

we have $x_{-m_k}$ converges to $x_\alpha$ and $x_{n_k}$ converges to $x_\omega$. Thus the same control established for (9.5.44) results in the limits

$$\delta_\omega^k \longrightarrow 0 \quad \text{and} \quad \delta_\alpha^k \longrightarrow 0 \quad \text{as } k \to \infty \tag{9.5.49}$$

As a result, $\delta(u, q_{|[-m_k, n_k]})$ converges to $\delta_\infty$ given by

$$\delta_\infty = [h_0^\omega(x_\omega) - u_\omega(0, q_\omega)] - [h_0^\alpha(x_\alpha) - u_\alpha(0, q_\alpha)] \tag{9.5.50}$$

We claim the following lemma

**Lemma 9.5.15.** *We have*

$$u_\alpha(0, q_\alpha) = h_0^\alpha(x_\alpha) \quad \textit{and} \quad u_\omega(0, q_\omega) = h_0^\omega(x_\omega) \tag{9.5.51}$$

This lemma leads to the nullity of $\delta_\infty$ and consequently to

$$0 \le \delta(u, q_{|[a,b]}) \le \liminf_k \delta(u, q_{|[-m_k, n_k]}) = \delta_\infty = 0 \tag{9.5.52}$$

which implies the calibration. □

*Proof of Lemma 9.5.15.* We only prove the equality for $x_\alpha = (q_\alpha, p_\alpha)$. Recall from Proposition 9.5.12 that $q_\alpha(t)$ is $u_\alpha$-calibrated. Then we can apply Proposition 9.5.8 at time zero to get that $u_\omega(t, q)$ is differentiable at $(0, q_\alpha)$ and

$$d_q u_\alpha(0, q_\alpha) = \partial_v L(0, q_\alpha, \dot{q_\alpha}(0)) = p_\alpha \quad \text{and} \quad \mathscr{H}(0, \partial_t u_\alpha(0, q_\alpha), q_\alpha, p_\alpha) = 0 \tag{9.5.53}$$

This implies that

$$du_\alpha(0, q_\alpha) = (\partial_t u_\alpha(0, q_\alpha), d_q u_\alpha(0, q_\alpha)) \in \{\mathscr{H} = 0\} \cap T^*_{(0, q_\alpha)}(\mathbb{R} \times M) \tag{9.5.54}$$

Moreover, We know from Lemma 9.5.9 that

$$du_\alpha(0, q_\alpha) = (\partial_t u_\alpha(0, q_\alpha), d_q u_\alpha(0, q_\alpha)) \in C_{u_\alpha}(0, q_\alpha) \subset \{\mathscr{H} \le 0\} \cap T^*_{(0, q_\alpha)}(\mathbb{R} \times M)$$

where the last inclusion, due to the strict convexity of the set $\{\mathscr{H} = 0\} \cap T^*_{(0,q_\alpha)}(\mathbb{R} \times M)$, has been established in the proof of Proposition 9.5.8. This strict convexity tells that $\{\mathscr{H} = 0\} \cap T^*_{(0,q_\alpha)}(\mathbb{R} \times M)$ is the set of extremal points of its convex hull $\{\mathscr{H} \le 0\} \cap T^*_{(0,q_\alpha)}(\mathbb{R} \times M)$.

Thus, we infer from the inclusion (9.5.54) that $du_\alpha(0,q_\alpha)$ is an extremal point of the convex set $C_{u_\alpha}(0,q_\alpha)$, or in other words that $du_\alpha(0,q_\alpha) \in K_{u_\alpha}(0,q_\alpha)$. By definition of the set $K_{u_\alpha}(0,q_\alpha)$, there exists a sequence $(t_n,q_n)$ in the set $\mathcal{U}_\alpha$ introduced in Proposition 9.4.20 converging to $(0,q_\alpha)$ such that $du_\alpha(t_n,q_n)$ converges to $du_\alpha(0,q_\alpha) = (\partial_t u_\alpha(0,q_\alpha), p_\alpha)$.

By definition of the set $\mathcal{U}_\alpha$, we have

$$\big(t_n, \partial_t u_\alpha(t_n,q_n), q_n, d_q u_\alpha(t_n,q_n)\big) \in \mathscr{L}_\alpha \quad \text{and} \quad u_\alpha(t_n,q_n) = \hbar_\alpha\big(t_n, \partial_t u_\alpha(t_n,q_n), q_n, d_q u_\alpha(t_n,q_n)\big)$$

We conclude that $\big(0, \partial_t u_\alpha(0,q_\alpha), q_\alpha, p_\alpha\big) \in \mathscr{L}_\alpha$ and, by continuity of the maps $u_\alpha$ and $\hbar_\alpha$, that

$$\begin{aligned} u_\alpha(0,q_\alpha) &= \lim_n u_\alpha(t_n,q_n) = \lim_n \hbar_\alpha\big(t_n, \partial_t u_\alpha(t_n,q_n), q_n, d_q u_\alpha(t_n,q_n)\big) \\ &= \hbar_\alpha\big(0, \partial_t u_\alpha(0,q_\alpha), q_\alpha, p_\alpha\big) = h_0^\alpha(q_\alpha,p_\alpha) = h_0^\alpha(x_\alpha) \end{aligned}$$

□

### 9.5.5 Proofs of the Main Results

We have seen in Propositions 9.5.12 and 9.5.14 that reduced asymptotic complexity in both positive and negative times implies that the extended Lagrangian submanifold $\mathscr{L}$ is foliated by calibrated curves. By applying Fathi's Theorem 9.5.8 on calibrated curves, it will follow that $\mathscr{L}$ is a graph over the zero section of $T^*(\mathbb{R}\times M)$.

*Proof of Theorem 9.1.5.* Let $t \in \mathbb{R}$ be a fixed time. For all $x = (q,p)$ in $\mathcal{L}_t$ and $x(s) = (q(s),p(s)) = \phi_H^{t,s}(q,p)$, we know from Proposition 9.5.14 that $q(s)$ is calibrated by $u$. Hence, we infer from Proposition 9.5.8 that $u$ is differentiable at $(t,q)$ and $d_q u(t,q) = \partial_v L(t,q,\dot q(t)) = p$. Thus, if we denote by $\mathcal{G}(du_t)$ the graph of $du_t = d_q u(t,\cdot)$ in $T^*M$, we get the inclusion $\mathcal{L}_t \subset \mathcal{G}(du_t)$. And since the projection $\mathcal{L}_t \to M$ is onto, we conclude that $\mathcal{L}_t = \mathcal{G}(du_t)$ is a graph over $0_{T^*M}$.

In order to obtain regularity, using the notation of Proposition 9.5.8, we proved that for all $\varepsilon > 0$, we have $A_{\varepsilon,u} = \mathbb{R}\times M$. Hence, $du$ is locally Lipschitz on $\mathbb{R}\times M$. And since the Lagrangian submanifolds $\mathcal{L}_t$ are $C^1$ regular, we conclude that they are $C^1$ graphs over the zero section $0_{T^*M}$. □

*Proof of Corollary 9.1.9.* The proof of this result is based on tools coming from the weak-KAM theory. We will show that the graph selector $u(t,x)$ is a recurrent (viscosity) solution of the Hamiltonian-Jacobi equation (5.0.1) associated with the Hamiltonian $H$. We present the concepts in the Appendix B.

All the proofs of this chapter remain unchanged by adding a constant to the Hamiltonian $H$. Hence, up to considering $H - \alpha_0$, we can assume that the Mañé critical value $\alpha_0$ is null. We consider the positive Lax-Oleinik operator $\mathcal{T}_+^{s,t}$ and we show that for all times

$s > t$, $\mathcal{T}_+^{s,t}u(s) = u(t)$. Fix a point $q$ in $M$ and two times $s > t$. We saw in Proposition 9.5.14 that there exists a $u$-calibrated curve $q(\tau)$ with $q(t) = q$. Then, we have

$$u(t,q) = u(s,q(s)) - h^{t,s}(q,q(s)) \le \mathcal{T}_+^{s,t}u(s)(q) = \sup_{q' \in M}\left\{u(s,q') - h^{t,s}(q,q')\right\}$$

Moreover, we know from Proposition 9.5.4 that for any point $q'$ in $M$, we have

$$u(s,q') - u(t,q) \le h^{s,t}(q,q')$$

yielding

$$\mathcal{T}_+^{s,t}u(s)(q) = \sup_{q' \in M}\left\{u(s,q') - h^{t,s}(q,q')\right\} \le u(t,q)$$

We obtained the equality $u(t) = \mathcal{T}_+^{s,t}u(s)$.

Fix a point $q_0$ in $M$. By Proposition B.0.4 the family $\left(\mathcal{T}_+^{s,t}u(s)\right)_{t \le s-1}$ is uniformly bounded for the $C^0$ norm. Hence, we can assume up to extraction that the sequence $u(-m_n, q_0)$ converges to a limit $c_\alpha \in \mathbb{R}$. Adding a constant to $u_\alpha$, we can also assume that $u_\alpha(q_0) = c_\alpha$. Then, since we have already established that $\lim_n \operatorname{Osc}(u_{-m_n} - u_\alpha) = 0$, we obtain the convergence of $u_{-m_n}$ to $u_\alpha$ in the $C^0$-topology.

We further assume up to extraction that the sequence $m_{n+1} - m_n$ diverges to infinity. We consider the negative Lax-Oleinik operator $\mathcal{T}_-^{s,t}$ with $\mathcal{T}_-^t := \mathcal{T}_-^{0,t}$. Analogously to the above, the map $u$ verifies for all times $s < t$, $u(t) = \mathcal{T}_-^{s,t}u(s)$. Therefore, the non-expansiveness of the $\mathcal{T}_-^{s,t}$ given by Proposition B.0.2 leads to

$$\begin{aligned}
\|u(-m_{n+1} + m_n) - u(0)\|_\infty &= \|\mathcal{T}_-^{-m_{n+1}+m_n}u(0) - u(0)\|_\infty \\
&= \|\mathcal{T}_-^{m_n}u_{-m_{n+1}} - \mathcal{T}_-^{m_n}u_{-m_n}\|_\infty \\
&\le \|u_{-m_{n+1}} - u_{-m_n}\|_\infty \\
&\le \|u_{-m_{n+1}} - u_\alpha\|_\infty + \|u_\alpha - u_{-m_n}\|_\infty \longrightarrow 0 \quad \text{as } n \to \infty
\end{aligned}$$

This shows that $u$ is a recurrent map. In particular, it is possible to take $u_\alpha$ and $u_\omega$ to be equal to $u_0$. The definition of reduced asymptotic complexity provides the respective Hausdorff convergence of the graphs $\mathcal{L}_{-m_k}$ and $\mathcal{L}_{n_k}$ of $du_{-m_k}$ and $du_{n_k}$ to the graph $\mathcal{L}$ of $du_0 = du_\alpha = du_\omega$. This concludes the proof of the $C^0$-recurrence of the Lagrangian $\mathcal{L}$. $\square$

**Remark 9.5.16.** More generally, this Corollary is a direct consequence of Theorem 9.1.5 and the following Proposition

**Proposition 9.5.17.** *Let $u : \mathbb{R} \times M \to \mathbb{R}$ be a $C^1$ global solution of the Hamilton-Jacobi equation*

$$\partial_t u + H(t,q,d_q u(t,q)) = \alpha_0 \tag{9.5.55}$$

*where $\alpha_0$ is the critical Mañé value (introduced in (5.1.6)). Then $u$ is $C^1$-recurrent in positive and negative (integer) times.*

The proof of the $C^0$-recurrence is included in the proof of Corollary 9.1.9. However, the $C^1$-recurrence is more intricate and follows from a result by M-C.Arnaud [Arn05] which states that if $\mathcal{T}_\pm^{n_k} u$ converges in $C^0$-topology to a scalar map $v$, then the graph of $d\mathcal{T}_\pm^{n_k} u$ converges for the Hausdorff distance to the graph of $dv$.

# Annexe A

# The Fathi-Mather Example for the Non-Convergence of the Lax-Oleinik Semi-group

In Section 3.2, we presented, without proofs, an example of a Tonelli Hamiltonian for which the convergence of the Lax-Oleinik semigroup is not satisfied. This construction is inspired by ideas from Fathi and Mather [FM00], who developed a more general example that includes the case of the pendulum considered.

In what follows, we propose to explore in more detail Fathi's original ideas and then interpret their results in terms of areas and representation formulas. The objective here is to clarify the exposition of Section 3.2 by performing explicit calculations.

## A.1 The Construction of Fathi-Mather

Fathi and Mather work on the circle $M = \mathbb{T}^1$ and consider a Tonelli Hamiltonian $H' : \mathbb{T}^1 \times T^*M \to \mathbb{R}$ with a Tonelli Lagrangian $L' : \mathbb{T}^1 \times TM \to \mathbb{R}$.

Let $p/q \in \mathbb{Q}$ be an irreducible rational number, and let $\mathcal{M}'_{p/q} \subset \mathbb{T}^1 \times TM$ be the union of all minimizing periodic orbits of period $q$ and rotation number $p/q$. Aubry-Mather theory on cylinder twists (see Section 1.3 and [Ban88, Den87, Mat91]) asserts that the set $\mathcal{M}'_{p/q}$ is closed, non-empty, invariant under the Euler-Lagrange flow $\phi_L$, and is a Lipschitz graph over its projection on $\mathbb{R} \times 0_{TM}$.

Recall that Mather's alpha and beta functions defined in (7.2.3) and (1.4.4) are convex functions with superlinear growth. We consider the set $\mathcal{L}_\beta$ known as the Legendre trans-form associated to $\beta = \beta_{L'}$. For $h \in H_1(M,\mathbb{R})$, the set $\mathcal{L}_\beta(h)$ is a non-empty, convex, and compact subset of $H^1(M,\mathbb{R})$ defined by

$$\mathcal{L}_\beta(h) = \left\{c \in H^1(M,\mathbb{R}) \mid \beta_{L'}(h) + \alpha_{L'}(c) = \langle c, h\rangle\right\} \tag{A.1.1}$$

In the case of the circle $M = \mathbb{T}^1$, we have $H^1(\mathbb{T}^1,\mathbb{R}) \simeq H_1(\mathbb{T}^1,\mathbb{R}) \simeq \mathbb{R}$. The work of Bangert [Ban94] shows that

- If $\omega$ is irrational, then $\mathcal{L}_\beta(\omega)$ is reduced to a point.
- If $\omega = p/q$, then $\mathcal{L}_\beta(p/q)$ is reduced to a point if and only if $\Sigma_{p/q} := \pi(\mathcal{M}'_{p/q}) = \mathbb{T}^1 \times M$.

And according to the work of Mañé [Mn96], it is known that for a generic Lagrangian $L$, the invariant set $\mathcal{M}'_{p/q}$ consists of a unique orbit [1]. In this case, $\Sigma_{p/q}$ is homeomorphic to a circle, and according to Bangert's theorem, $\mathcal{L}_\beta(p/q) = [c_-, c_+]$ with $c_- < c_+$.

We consider the case of $q \geq 2$ and we fix a closed 1-form $\eta$ on $M$ such that the de Rham cohomology class $c := [\eta]$ satisfies $c_- < c < c_+$. Fathi and Mather assert that the Tonelli Lagrangian $L = L' - \eta$ provides an example for which the Lax-Oleinik semigroup does not converge. The theorem is stated as follows.

**Theorem A.1.1.** *There exists a point $x \in M = \mathbb{T}^1$ such that $h^n(x,x)$ does not converge as $n$ tends to $+\infty$.*

This is sufficient to show the non-convergence of the Lax-Oleinik semigroup. Indeed, the potential $h^t$ satisfies the relation $h^{n+1}(x,x) = \mathcal{T}^n h^1(x,\cdot)(x)$ (see Proposition 5.2.7).

**Outline of the Proof.** For each integer $0 \leq k < q$, they introduce the $(q,k)$-Peierls barrier $h^{q\infty+k}$, adapted for the study of the $q$-periodic curves of $\mathcal{M}'_{q/p}$.

$$h^{q\infty+k}(x,y) = \liminf_n h^{qn+k}(x,y) \tag{A.1.2}$$

This gives

$$h^\infty(x,y) = \min_{0\leq k<q-1} h^{q\infty+k}(x,y) \tag{A.1.3}$$

Let $(t,\tilde{x}(t))$ lbe the orbit of $\mathcal{M}'_{p/q} \subset \mathbb{T}^1 \times TM$, projected onto the orbit $(t, x(t))$ of $\Sigma_{p/q}$ and denote by $(x_i = x(i))_{0\leq i<q-1}$ the $q$ points of $\Sigma^0_{p/q} := \Sigma_{p/q} \cap (\{0\} \times M)$.

Their goal is to show that there exists an integer $k$ such that $h^{q\infty+k}(x_i,x_i) > 0$. Indeed, if this is verified, then by the properties of Peierls barriers cited in Proposition 5.2.14 and

1. This holds for our current definition of $\mathcal{M}'_{p/q}$ comme unions des courbes périodiques minimisantes, as unions of minimizing periodic curves, and not the full Aubry-Mather set $\mathcal{M}_{p/q}$ of rotation number $p/q$, which also contains heteroclinic curves.

since $x_i$ belongs to the Mather set, we would have

$$0 = h^{\infty}(x_i, x_i) = \liminf_n h^n(x_i, x_i) < h^{q\infty+k}(x_i, x_i) = \liminf_n h^{qn+k}(x_i, x_i)$$

proving the non-convergence of $h^n(x_i, x_i)$.

Let us show the existence of $k$ such that $h^{q\infty+k}(x_i, x_i) > 0$. Let $u$ be a weak KAM solution. Then, since the curve $(t, x(t))$ belongs to the Mather set $\mathcal{M} \subset \mathcal{A}$ and by the definition (2.1.11) of $\mathcal{A}$, we know that $x(t)$ is a static curve which, in particular, is $u$-calibrated. Thus, for all integers $0 \le i, j < q - 1$ and $n \ge 1$

$$u(x_j) - u(x_i) = u(qn + j, x(qn + j)) - u(i, x(i)) = h^{qn+j-i}(x_i, x_j)$$

and taking the liminf as $n$ tends to infinity, we get

$$u(x_j) - u(x_i) = h^{q\infty+j-i}(x_i, x_j)$$

In particular, we find

$$h^{q\infty+j-i}(x_i, x_j) + h^{q\infty+i-j}(x_j, x_i) = u(x_j) - u(x_i) + u(x_i) - u(x_j) = 0$$

Since the $q$-barrier $h^{q\infty}$ also satisfies the triangular inequality (5.2.28), we apply this inequality twice to obtain

$$\begin{aligned} h^{q\infty}(x_i, x_j) &\le h^{q\infty+i-j}(x_i, x_i) + h^{q\infty+j-i}(x_i, x_j) \\ &\le h^{q\infty}(x_i, x_j) + h^{q\infty+i-j}(x_j, x_i) + h^{q\infty+j-i}(x_i, x_j) \\ &= h^{q\infty}(x_i, x_j) \end{aligned}$$

Thus, we have equality everywhere, which gives

$$h^{q\infty}(x_i, x_j) = h^{q\infty+i-j}(x_i, x_i) + h^{q\infty+j-i}(x_i, x_j)$$

Similarly, we show that

$$h^{q\infty}(x_j, x_i) = h^{q\infty+i-j}(x_j, x_i) + h^{q\infty+j-i}(x_i, x_i)$$

And by summing these two identities, we obtain

$$\begin{aligned} d_q(x_i, x_j) &:= h^{q\infty}(x_i, x_j) + h^{q\infty}(x_j, x_i) \\ &= h^{q\infty+i-j}(x_i, x_i) + h^{q\infty+j-i}(x_i, x_j) + h^{q\infty+i-j}(x_j, x_i) + h^{q\infty+j-i}(x_i, x_i) \qquad \text{(A.1.4)} \\ &= h^{q\infty+i-j}(x_i, x_i) + h^{q\infty+j-i}(x_i, x_i) \end{aligned}$$

Moreover, since $x_i \in \mathcal{A} = \{h^\infty(x,x) = 0\}$ and by the identity (A.1.3), we know that for any integer $k$

$$h^{q\infty+k}(x_i,x_i) \geq h^\infty(x_i,x_i) = 0$$

We deduce that if $d_q(x_i,x_j) > 0$, then either $h^{q\infty+i-j}(x_i,x_i) > 0$ or $h^{q\infty+j-i}(x_i,x_i) > 0$ (or both). It remains to evaluate $d_q(x_i,x_j)$ and show that it is strictly positive. This is the subject of the central lemma in Fathi-Mather's article, stated as follows.

**Lemma A.1.2.** *For the considered Lagrangian $L$, we have*

$$d_q(x_i,x_j) = \min\left(c_+ - c, c - c_-, \|\{pi/q\} - \{pj/q\}\|(c_+ - c_-)\right) \tag{A.1.5}$$

*where $\{\tau\} \in \mathbb{T}^1$ is the fractional part of $\tau \in \mathbb{R}$ and $\|\{\tau\}\| = \min\{|\tau - n|\ ;\ n \in \mathbb{R}\}$.*

Their proof, highly technical, relies on manipulating minimizing curves that realize $h^{q\infty}(x_i,x_j)$ and studying the dependence with respect to the cohomology class $c$.

We propose to prove this lemma in the particular case of the non-autonomous pendulum, bypassing technicalities and interpreting this formula as the area of well-chosen regions in phase space.

## A.2 Interpretation and Application to the Pendulum

We will demonstrate the formula from Lemma A.1.2 while examining what occurs for the $q$-non-autonomous pendulum with Hamiltonian

$$H(t,x,P) = \frac{1}{2}P^2 + \cos\left(2q\pi\left(x - \frac{pt}{q}\right)\right) \tag{A.2.1}$$

Note that for this Tonelli Hamiltonian, the point $x_0 = (0,0)$ is periodic with rotation number $p/q$, and its orbit is precisely $\mathcal{M}'_{p/q} = \Sigma_{p/q} = \{(pi/q, 0)\ ;\ i = 0,..,q-1\}$.

We fix $i = 0$ and aim to evaluate $d_q(x_0,x_j)$.

According to the work of M-C. Arnaud and M. Zavidovique on twists of the cylinder [AZ23], pour chacune des deux constantes $c\pm$, for each of the two constants $c_\pm$, there exists a unique, up to a constant, viscosity solution $u_{c_\pm}$ that is $q$-periodic and associated with the Lagrangian $L_{c_\pm} = L - c_\pm$. These solutions satisfy the inequality $u'_{c_-} \leq u'_{c_+}$ on $M$. In the case of the pendulum, their differentials correspond to the upper and lower segments of the critical energy level, as depicted in Figure (A.1a).

Moreover, in these critical cases where $c = c_\pm$, and for Hamiltonians of the form

$$H(t,x,P) = \frac{1}{2}P^2 + V(t,x)$$

S. Motonaga [Mot22] shows that the graphs of the differentials of the critical solutions $du_{c_\pm}$ are equal to the Aubry sets $\mathcal{A}_{c_\pm}$ associated to $L_{c_\pm}$. In particular, these solutions are $C^1$-regular and therefore are classical solutions of the Hamilton-Jacobi equation.

**Remark A.2.1.** This specific point is an obstacle to proving Lemma A.1.2 in its full generality and forces us to restrict to the case of the pendulum. It is not clear to me whether, if $c$ is an extremal point of Mather's alpha function $\alpha$, the associated Aubry set projects onto the entire base $\mathbb{T}^1$, which is equivalent to the existence of a classical solution to the Hamilton-Jacobi equation.

From the $C^1$ regularity of $u_{c_\pm}$, we deduce that they are also $q$-periodic solutions associated with the positive semi-group $\mathcal{T}_+$ introduced in Appendix B. Note that Contreras, Iturriaga, and Sánchez-Morgado [CISM13] show that a representation formula similar to (2.2.5) also holds for the positive semi-group $\mathcal{T}_+$, given for $q$-periodic solutions by

$$\begin{array}{rcl} \Psi_q^+ : \mathrm{Dom}(\mathbb{M}_q, h^{q\infty}) & \longrightarrow & \mathrm{Per}_q(\mathcal{T}) \\ \psi & \longmapsto & \sup\limits_{y\in\mathbb{M}_q} \{\psi(y) - h^{q\infty}(\cdot,y)\} \end{array}$$

Finally, for every $c \in [c_-, c_+]$, the $q$-periodic solutions $u_c$ associated to $L_c = L - c$ are those whose differential graph $\mathcal{G}(du_c)$ is invariant under $\phi_{L_c}^{-q}$ and which satisfy

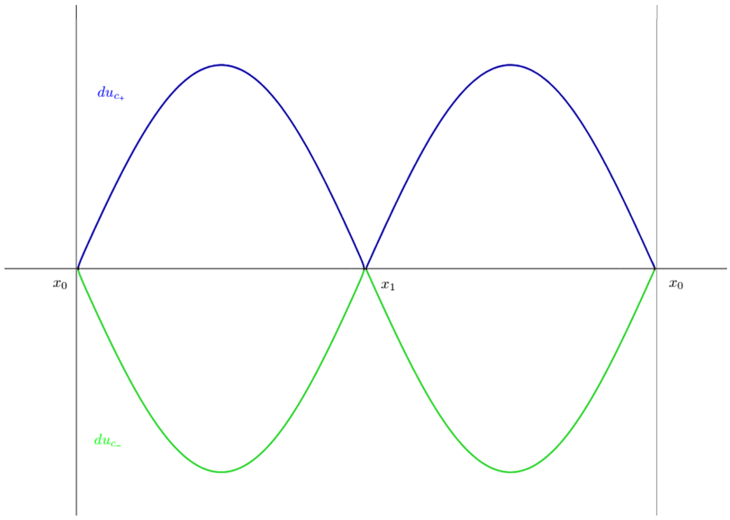
$$\mathcal{G}(c + du_c) \subset \mathcal{G}(c_+ + du_{c_+}) \cup \mathcal{G}(c_- + du_{c_-}) \tag{A.2.2}$$

One of these solutions is represented for $c = \frac{c_+ + c_-}{2}$ in Figure A.1b.

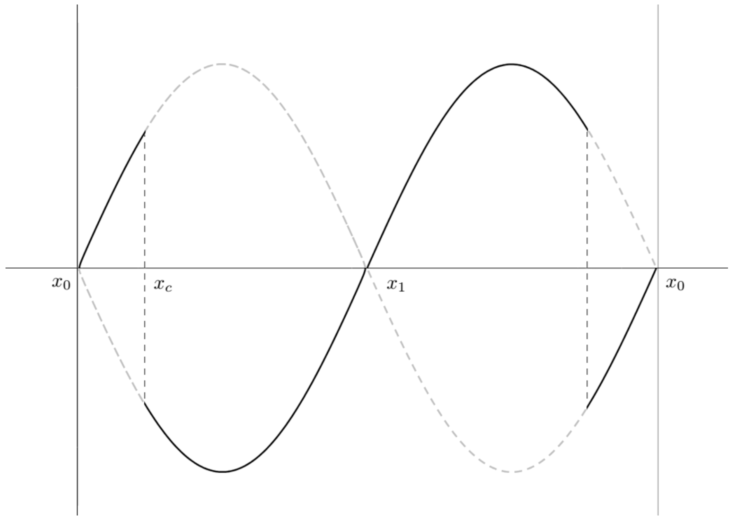
The representation formulas show that for any $x \in M$

$$\begin{aligned} h^{q\infty}(x_0,x) &= \sup_{v\in\mathrm{Per}_q(\mathcal{T})} v(x) - v(x_0) \\ -h^{q\infty}(x,x_0) &= \inf_{v\in\mathrm{Per}_q(\mathcal{T}_+)} v(x) - v(x_0) \end{aligned} \tag{A.2.3}$$

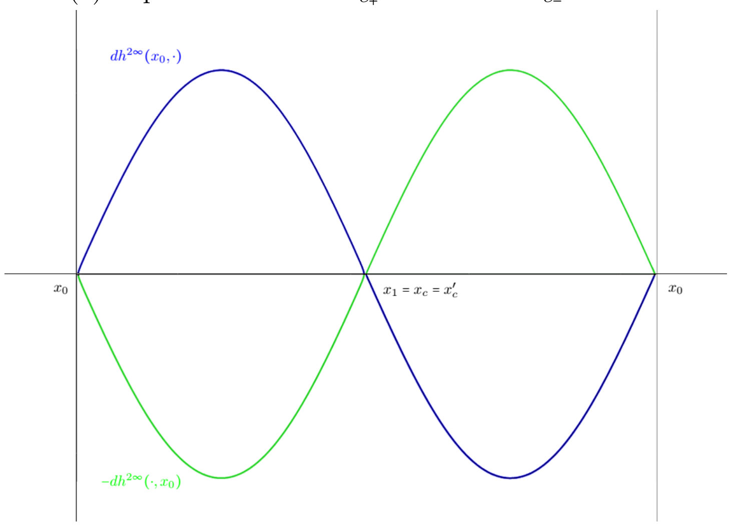
Thus, by considering these supremum and infimum for the possible weak KAM solu-



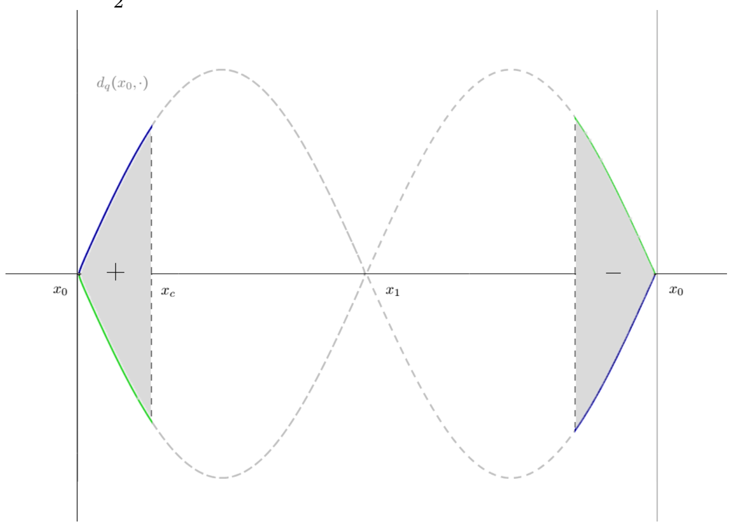

(a) Représentation de $du_{c_+}$ en bleu et $du_{c_-}$ en vert.




(b) Représentation de la différentielle d'une solution 2-périodique pour $c = \frac{c_+ + c_-}{2}$.




(c) Représentation de $dh^{2\infty}(x_0, \cdot)$ en bleu et $-dh^{2\infty}(\cdot, x_0)$ en vert avec $x_c = x'_c$.




(d) Représentation en aires de $d_q(x_0, \cdot)$ dans le cas $x_c < x'_c$. L'aire à gauche est comptée positivement, et l'aire à droite négativement.

FIGURE A.1 – Représentation du pendule avec $p/q = 1/2$

tions described earlier, we deduce the existence of two points $x_c$ and $x'_c$ in $M$ such that

$$\begin{aligned} c + dh^{q\infty}(x_0,\cdot)(x) &= \begin{cases} c_+ + du_{c_+}(x) & \text{si } x \in [0, x_c] \\ c_- + du_{c_-}(x) & \text{si } x \in [x_c, 1] \end{cases} \\ c - dh^{q\infty}(\cdot, x_0)(x) &= \begin{cases} c_- + du_{c_-}(x) & \text{si } x \in [0, x'_c] \\ c_+ + du_{c_+}(x) & \text{si } x \in [x'_c, 1] \end{cases} \end{aligned} \tag{A.2.4}$$

These differentials are illustrated in Figure A.1c.

Integrating the expressions in (A.2.4) over $M = \mathbb{T}^1$ yields

$$\begin{aligned} c &= \int_0^{x_c} (c_+ + du_{c_+}(x))\, dx + \int_{x_c}^1 (c_- + du_{c_-}(x))\, dx \\ &= \int_0^{x'_c} (c_- + du_{c_-}(x))\, dx + \int_{x'_c}^1 (c_+ + du_{c_+}(x))\, dx \end{aligned} \tag{A.2.5}$$

We also deduce from (A.2.4) the differential $d_q(x_0, \cdot)$ as follows. Assuming $x_c \le x'_c$ in $[0,1]$, we get

$$\begin{aligned} d(d_q(x_0,\cdot))(x) &= [c + dh^{q\infty}(x_0,\cdot)(x)] - [c - dh^{q\infty}(\cdot, x_0)(x)] \\ &= \begin{cases} c_+ + du_{c_+}(x) - c_- - du_{c_-}(x) & \text{si } x \in [0, x_c] \\ 0 & \text{si } x \in [x_c, x'_c] \\ c_- + du_{c_-}(x) - c_+ - du_{c_+}(x) & \text{si } x \in [x'_c, 1] \end{cases} \end{aligned} \tag{A.2.6}$$

Hence, $d_q(x_0, \cdot)$ is constant on $[x_c, x'_c]$ and equal to

$$\begin{aligned} d_q(x_0, x_c) &= \int_0^{x_c} d(d_q(x_0,\cdot))(x)\, dx = \int_0^{x_c} (c_+ + du_{c_+}(x))\, dx - \int_0^{x_c} (c_- + du_{c_-}(x))\, dx \\ &= \int_0^{x_c} (c_+ + du_{c_+}(x))\, dx + \int_{x_c}^1 (c_- + du_{c_-}(x))\, dx - c_- \\ &= c - c_- \end{aligned}$$

A simple way to see this is by considering areas. We have $d_q(x_0, x_0) = 0$ and $d(d_q(x_0,\cdot))(x) = dh^{q\infty}(x_0,\cdot)(x) - (-dh^{q\infty}(\cdot, x_0)(x))$. Thus, we deduce that $d_q(x_0, x_0)$ is the area between the graphs of $dh^{q\infty}(x_0,\cdot)$ and $-dh^{q\infty}(\cdot, x_0)$, as shown in Figure A.1d.

Let us note that in this case, we also have from identity (A.2.5) that

$$c_+ - c = c_+ - \int_0^{x'_c} (c_- + du_{c_-}(x))\, dx - \int_{x'_c}^1 (c_+ + du_{c_+}(x))\, dx$$

$$= (c_+ - c_-)x'_c + \int_0^{x'_c} (du_{c_+}(x) - du_{c_-}(x))\, dx$$

and

$$c - c_- = (c_+ - c_-)x_c + \int_0^{x_c} (du_{c_+}(x) - du_{c_-}(x))\, dx$$

yielding

$$(c_+ - c) - (c - c_-) = (c_+ - c_-)(x'_c - x_c) + \int_{x_c}^{x'_c} (du_{c_+}(x) - du_{c_-}(x))\, dx \geq 0 \tag{A.2.7}$$

Hence, we obtain the inequality

$$c - c_- \leq c_+ - c \tag{A.2.8}$$

Analogously, we show that for $x_c \geq x'_c$ in $[0,1]$,

$$d(d_q(x_0,\cdot))(x) = \begin{cases} c_+ + du_{c_+}(x) - c_- - du_{c_-}(x) & \text{si } x \in [0, x'_c] \\ 0 & \text{si } x \in [x'_c, x_c] \\ c_- + du_{c_-}(x) - c_+ - du_{c_+}(x) & \text{si } x \in [x_c, 1] \end{cases} \tag{A.2.9}$$

with

$$d_q(x_0, x'_c) = c_+ - c \tag{A.2.10}$$

and

$$c - c_- \geq c_+ - c \tag{A.2.11}$$

Let us evaluate $d_q(x_0, x_j)$. We set $y_c = \min(x_c, x'_c)$ and $z_c = \max(x_c, x'_c)$. Then, we have

$$d_q(x_0, x_j) = \begin{cases} \int_{x_0}^{x_j} (c_+ - c_- + du_{c_+}(x) - du_{c_-}(x))\, dx & \text{si } x_j \in [0, y_c] \\ \min(c_+ - c, c - c_-) & \text{si } x_j \in [y_c, z_c] \\ \int_{x_j}^{x_0+1} (c_+ - c_- + du_{c_+}(x) - du_{c_-}(x))\, dx & \text{si } x_j \in [z_c, 1] \end{cases} \tag{A.2.12}$$

An argument by Fathi and Mather shows that the application $d_q(x_i, x_j)$ depends only on $\|\{pi/q\} - \{pj/q\}\|$. We reorder the points $x_j$ in increasing order as $x^k$ in $[0,1]$ with $x^0 = x_0$. Since $p$ and $q$ are coprime, we can verify that $k = q\|\{pj/q\}\| = q\|\{pj/q\} - \{p.0/q\}\|$.

By taking $c = \frac{c_+ + c_-}{2}$, we get $x_c = x'_c$ and the formula for $d_q$ becomes

$$d_q(x^k, x^l) = \int_{x^k}^{x^l} (c_+ - c_- + du_{c_+}(x) - du_{c_-}(x))\, dx$$

Thus, we obtain

$$\begin{aligned} c_+ - c_- &= \int_{x_0}^{x_0+1} (c_+ - c_- + du_{c_+}(x) - du_{c_-}(x))\, dx \\ &= \sum_{k=0}^{q-1} d_q(x^k, x^{k+1}) = q d_q(x^0, x^1) \end{aligned}$$

and if for $0 \leq k \leq l \leq q-1$, the points $x^k$ and $x^l$ are associated to $x_i$ and $x_j$, we get

$$d_q(x_i, x_j) = d_q(x^k, x^l) = (l-k).d_q(x^0, x^1) = \frac{l-k}{q}(c_+ - c_-) = \|\{pi/q\} - \{pj/q\}\|(c_+ - c_-)$$

We showed that

$$d_q(x_0, x_j) = \begin{cases} \|\{pi/q\} - \{pj/q\}\|(c_+ - c_-) & \text{si } x_j \in [0, y_c] \cup [z_c, 1] \\ \min(c_+ - c, c - c_-) & \text{si } x_j \in [y_c, z_c] \end{cases} \tag{A.2.13}$$

Finally, since $d_q(x_0, x_j)$ is increasing on $[0, y_c]$ and decreasing on $[z_c, 1]$,we ultimately obtain formula (A.1.5) of Lemma A.1.2.

# Annexe B

# The Positive Lax-Oleinik Operator

This appendix is devoted to defining the positive Lax-Oleinik operator and discussing two of its basic properties that will be needed in the proof of Corollary 9.1.9. The negative operator generates viscosity solutions of the Hamilton-Jacobi equation, while the positive operator introduces new objects. It is defined as follows.

**Definition B.0.1.** Fix two times $s < t$,

1. The *positive Lax-Oleinik operator* $\mathcal{T}_{+,0}^{t,s} : \mathcal{C}^0(M,\mathbb{R}) \to \mathcal{C}^0(M,\mathbb{R})$ is defined by

$$\begin{aligned}\mathcal{T}_{+,0}^{t,s} u_0(x) &= \sup_{\substack{\gamma : [s,t] \to M \\ s \mapsto x}} \left\{ u_0(\gamma(t)) - \int_s^t L(\tau, \gamma(\tau), \dot{\gamma}(\tau))\, d\tau \right\} \\ &= \sup_{y \in M} \left\{ u_0(y) - h_0^{s,t}(x,y) \right\}\end{aligned} \tag{B.0.1}$$

2. The *full positive Lax-Oleinik operator* $\mathcal{T}_{+}^{t,s} : \mathcal{C}^0(M,\mathbb{R}) \to \mathcal{C}^0(M,\mathbb{R})$ is defined by

$$\mathcal{T}_{+}^{t,s} u_0 = \mathcal{T}_{+,0}^{t,s} u_0 - \alpha_0.(s-t) \tag{B.0.2}$$

We omit the notation of the first time whenever it is null, i.e $\mathcal{T}_+^t := \mathcal{T}_+^{0,t}$ and we set $\mathcal{T}_+ := \mathcal{T}_+^1$.

As mentioned above, the negative Lax-Oleinik operator $\mathcal{T}_- := \mathcal{T}$ introduced in Definition 5.1.3 generates what is called the viscosity solutions $u(t,x) = \mathcal{T}_-^{0,t} u_0(x)$ of the Hamilton-Jacobi equation

$$\partial_t u + H(t, x, d_x u) = \alpha_0 \tag{B.0.3}$$

The positive Lax-Oleinik operator introduces a new type of weak solutions that were unknown before the development of A. Fathi's weak-KAM theory.

We present the non-expansiveness property which proof is analogous to the non-expansiveness of the negative Lax-Oleinik semi-group $\mathcal{T}_-$ seen in Proposition 6.1.1.

**Proposition B.0.2.** *Fix two times $s < t$. The Lax-Oleinik operators $\mathcal{T}_-^{s,t}$ and $\mathcal{T}_+^{t,s}$ are non-expansive, i.e for all continuous scalar maps $u$ and $v$ in $\mathcal{C}(M,\mathbb{R})$, we have*

$$\|\mathcal{T}_-^{s,t}u - \mathcal{T}_-^{s,t}v\|_\infty \leq \|u - v\|_\infty \quad \textit{and} \quad \|\mathcal{T}_+^{t,s}u - \mathcal{T}_-^{t,s}v\|_\infty \leq \|u - v\|_\infty \tag{B.0.4}$$

Elements $u \in \mathrm{Fix}(\mathcal{T}_+)$, fixed by the positive Lax-Oleinik operator $\mathcal{T}+$, are called *positive weak-KAM solution* of the Hamilton-Jacobi equation 5.0.1. Similarly to the negative weak-KAM solutions in $\mathrm{Fix}(\mathcal{T}_-)$, examples of positive weak-KAM solutions can be expressed using the Peierls barrier $h^\infty$ (see Section 5.2.3), and the set $\mathrm{Fix}(\mathcal{T}_+)$ can be described by a representation formula detailed in [CISM13]. Here, we only give the specific expression of positive weak-KAM solutions given by the Peierls barrier $h^\infty$.

**Proposition B.0.3.** *Fix a point $y \in M$. The map $u(t,x) = -h^{t,\infty}(x,y) + \alpha_0.t$ is a positive weak-KAM solution of the Hamilton-Jacobi equation. More precisely, for all times $s < t$, $\mathcal{T}_+^{t,s}u(t,\cdot) = u(s,\cdot)$ and $\mathcal{T}_+^{0,-1}u(0,\cdot) = u(0,\cdot)$.*

*Proof.* Let $v(t,x) = -h^{t,\infty}(x,y)$. We claim that for all times $s < t$, $\mathcal{T}_+^{t,s}v(t,\cdot) = v(s,\cdot)$. Indeed, we prove it establishing a double inequality.

Fix two times $s < t$ and a point $x \in M$. Let $k_n$ be an increasing sequence of integers such that $h^{s,\infty}(x,y) = \lim_n h^{s,k_n}(x,y)$. For every integer $n$, the compactness of $M$ ensures the existence of a point $z_n \in M$ realizing the following infimum

$$h^{s,t}(x,z_n) + h^{t,k_n}(z_n,y) = \inf_{z\in M}\{h^{s,t}(x,z) + h^{t,k_n}(z,y)\}$$

Moreover, it is easy to see that

$$\inf_{z\in M}\{h^{s,t}(x,z) + h^{t,k_n}(z,y)\} = h^{s,k_n}(x,y)$$

Hence, after assuming, up to extraction, that the sequence $z_n$ converges to a point $z \in M$, we deduce that

$$\begin{aligned} h^{s,\infty}(x,y) &= \lim_n h^{s,k_n}(x,y) = \lim_n h^{s,t}(x,z_n) + \liminf_n h^{t,k_n}(z_n,y) \\ &= h^{s,t}(x,z) + \liminf_n h^{t,k_n}(z_n,y) \\ &\geq h^{s,t}(x,z) + h^{t,\infty}(z,y) \end{aligned}$$

where we mainly used properties of Peierls barrier that can be found in Proposition 5.2.14. This yields a first inequality

$$\begin{aligned} v(s,x) = -h^{s,\infty}(x,y) &\leq -h^{t,\infty}(z,y) - h^{s,t}(x,z) \\ &\leq \mathcal{T}_+^{t,s}v(t)(x) = \sup_{z'\in M}\big\{ -h^{t,\infty}(z',y) - h^{s,t}(x,z')\big\} \end{aligned}$$

We now establish the inverse inequality. We know that for any point $z$ of $M$ and any large integer $n > s$,

$$h^{s,n}(x,y) \leq h^{s,t}(x,z) + h^{t,n}(z,y)$$

Taking the liminf on $n$, we get

$$h^{s,\infty}(x,y) \leq h^{s,t}(x,z) + h^{t,\infty}(z,y)$$

which holds for all points $z$ of $M$. Changing the sign and taking the supremum on $z$ gives the desired inequality

$$v(s,x) = -h^{s,\infty}(x,y) \geq \mathcal{T}_+^{t,s} v(t)(x) = \sup_{z \in M} \left\{ -h^{t,\infty}(z,y) - h^{s,t}(x,z) \right\}$$

We have shown that $\mathcal{T}_+^{t,s} v(t,\cdot) = v(s,\cdot)$. Let us establish the desired properties on the map $u(t,x)$. We have

$$\begin{aligned} \mathcal{T}_+^{t,s} u(t,\cdot) &= \mathcal{T}_+^{t,s} u(t,\cdot) - \alpha_0.(t-s) = \mathcal{T}_+^{t,s}\big(u(t,\cdot) - \alpha_0.t\big) + \alpha_0.s \\ &= \mathcal{T}_+^{t,s} v(t,\cdot) + \alpha_0.s = v(s,\cdot) + \alpha_0.s = u(s,\cdot) \end{aligned}$$

Additionally, we have

$$v(t+1,x) = -h^{t+1,\infty}(x,y) = -\liminf_n h^{t+1,n}(x,y) = -\liminf_n h^{t,n-1}(x,y) = -h^{t,\infty}(x,y) = v(t,x)$$

where we used the time periodicity of the Lagrangian $L$. This yields

$$\mathcal{T}_+^{0,-1} u(0,\cdot) = \mathcal{T}_{+,0}^{0,-1} u(t,\cdot) - \alpha_0 = u(-1,\cdot) - \alpha_0 = v(-1,\cdot) = v(0,\cdot) = u(0,\cdot)$$

□

**Proposition B.0.4.** *For any time $t \in \mathbb{R}$ and any scalar map $u \in \mathcal{C}^0(M,\mathbb{R})$, the family of maps $\left(\mathcal{T}_+^{t,s} u = \mathcal{T}_+^{t,s} u + \alpha_0.(t-s)\right)_{s<t}$ is uniformly bounded in the $C^0$ topology.*

*Proof.* This is mainly due to the non-expansiveness of the Lax-Oleinik operator $\mathcal{T}_+^{t,s}$ and to the existence of positive weak-KAM solutions. Fix a time $t \in \mathbb{R}$ and a scalar map $u \in \mathcal{C}^0(M,\mathbb{R})$ and let $w(\tau,x) = -h^{\tau,\infty}(x,y) + \alpha_0.\tau$ be a positive weak solution with initial data $w_0 = w(0,\cdot)$, introduced in Proposition B.0.3. The Proposition B.0.2 yields for all time $s \leq t$,

$$\|\mathcal{T}_+^{t,s} u - \mathcal{T}_+^{t,s}\mathcal{T}_+^{\lceil t\rceil,t} w_0\|_\infty \leq \|u - \mathcal{T}_+^{\lceil t\rceil,t} w_0\|_\infty$$

where $\lceil\cdot\rceil$ stand for the ceil map. Besides, we know from the definition of positive weak-KAM solutions that

$$\mathcal{T}_+^{t,s}\mathcal{T}_+^{\lceil t\rceil,t} w_0 = \mathcal{T}_+^{\lceil t\rceil,s} w_0 = \mathcal{T}_+^{\lceil s\rceil,s}\mathcal{T}_+^{\lceil t\rceil,\lceil s\rceil} w_0 = \mathcal{T}_+^{\lceil s\rceil,s} w_0 = \mathcal{T}_+^{0,s-\lceil s\rceil} w_0$$

Therefore, we obtain a uniform bound using the continuity in time $\tau$ of $\mathcal{T}_+^{\tau} w_0$ (see Proposition 5.1.13) as follows

$$\begin{aligned}
\|\mathcal{T}_+^{t,s} u\|_\infty &\leq \|u - \mathcal{T}_+^{\lceil t\rceil,t} w_0\|_\infty + \|\mathcal{T}_+^{t,s}\mathcal{T}_+^{\lceil t\rceil,t} w_0\|_\infty \\
&= \|u - \mathcal{T}_+^{\lceil t\rceil,t} w_0\|_\infty + \|\mathcal{T}_+^{0,s-\lceil s\rceil} w_0\|_\infty \\
&\leq \|u - \mathcal{T}_+^{\lceil t\rceil,t} w_0\|_\infty + \sup_{\tau\in[-1,0]} \|\mathcal{T}_+^{0,\tau} w_0\|_\infty < +\infty
\end{aligned}$$

□

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